Time dispersion in quantum mechanics

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Abstract

In quantum mechanics the time dimension is treated as a parameter, while the three space dimensions are treated as observables. This assumption is both untested and inconsistent with relativity. From dimensional analysis, we expect quantum effects along the time axis to be of order an attosecond. Such effects are not ruled out by current experiments. But they are large enough to be detected with current technology, if sufficiently specific predictions can be made. To supply such we use path integrals. The only change required is to generalize the usual three dimensional paths to four. We predict a large variety of testable effects. The principal effects are additional dispersion in time and full equivalence of the time/energy uncertainty principle to the space/momentum one. Additional effects include interference, diffraction, and entanglement in time. The usual ultraviolet divergences do not appear: they are suppressed by a combination of dispersion in time and entanglement in time. The approach here has no free parameters; it is therefore falsifiable. As it treats time and space with complete symmetry and does not suffer from the ultraviolet divergences, it may provide a useful starting point for attacks on quantum gravity.

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“Wheeler's often unconventional vision of nature was grounded in reality through the principle of radical conservatism, which he acquired from Niels Bohr: Be conservative by sticking to well-established

physical principles, but probe them by exposing their most radical conclusions." – Kip Thorne Thorne [2009].

"You can have as much junk in the guess as you like, provided that the consequences can be compared with experiment." – Richard Feynman Feynman [1965]

1 Introduction

In relativity, time and space enter on a basis of formal equivalence. In special relativity, the time and space coordinates rotate into each other under Lorentz transformations. In general relativity, the time and the radial coordinate change places at the Schwarzschild radius (for instance in Adler et al. [1965]). In wormholes and other exotic solutions to general relativity, time can even curve back on itself Gödel [1949], Thorne [1994].

But in quantum mechanics "time is a parameter not an operator" (Hilgevoord Hilgevoord [1996, 1998]). This is clear in the Schrödinger equation:

$$\imath \frac{d}{d\tau} \psi_\tau (\vec{x}) = \hat{H} \psi_\tau (\vec{x}) \tag{1}$$

Here the wave function is indexed by time: if we know the wave function at time τ we can use this equation to compute the wave function at time $\tau + \epsilon$. The wave function has in general non-zero dispersion in space, but is always treated as having zero dispersion in time. This would appear to be inconsistent with special relativity.
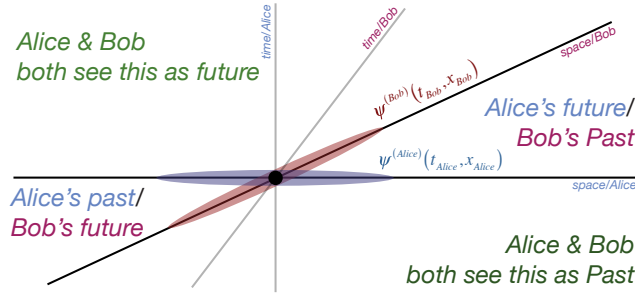
Figure 1: Wave functions for Alice and Bob

Consider Alice in her laboratory, her co-worker Bob jetting around like a fusion powered mosquito. Both are studying the same system but with respective wave functions:

$$\psi^{(Alice)}\left(t_{Alice}, x_{Alice}\right)$$
$$\psi^{(Bob)}\left(t_{Bob}, x_{Bob}\right) \tag{2}$$

Alice and Bob center their respective wave functions on the particle:

$$\langle t_{Alice}\rangle = \langle t_{Bob}\rangle = 0$$
$$\langle x_{Alice}\rangle = \langle x_{Bob}\rangle = 0 \tag{3}$$

These are distinct wave functions but give the same predictions for all observables. And do so to an extremely high degree of reliability.

But at Alice's time zero, Bob's wave function extends into her past and future. And at Bob's time zero her wave function extends into his past and future.

There are at least two problems here.

One is that in quantum mechanics there is a fairly strict "plane of the present". The quantum mechanical wave function is non-localized in space but is strictly localized in time. What if Alice decided to work with Bob's wave function, rather than her own? She will get by hypothesis all the same predictions, but will be using a wave function that from her point of view slops into past and future.

The other is that from the point of view of special relativity, there should not be a strict "plane of the present" in the first place. We should be able to rotate between the four dimensional references frames of Alice and Bob as easily as we rotate between references frames for the three space dimensions.

What happens if we shift to four dimensional wave functions?

$$\psi\left(\vec{x}\right) \rightarrow \psi\left(t, \vec{x}\right) \tag{4}$$

Assume the coordinate systems for Alice and Bob are related by a Lorentz transformation Λ:

$$x^{(Bob)} = \Lambda x^{(Alice)} \tag{5}$$

Then their wave functions can be related by a Lorentz transformation of their coordinates:

$$\psi^{(Bob)}\left(x^{(Bob)}\right) = \psi^{(Alice)}\left(\Lambda x^{(Alice)}\right) \tag{6}$$

and matters are much more straightforward.

We make this our basic hypothesis: *the quantum mechanical wave function should be extended in the time direction on the same basis as it is extended along the three space dimensions.*

We are playing a "game of if" here: we will develop the push the idea as hard as we can and see what breaks. We are not going to argue that this is or is not true. We are going to look for experimental tests and then let the experimentalists decide the question.

There are two principal effects:

1. Dispersion in time appears on same basis as dispersion in space. Physical wave functions are always a bit spread out in space; they will now also be a bit spread out in time.

2. The uncertainty principle for time/energy is treated on same basis as the uncertainty principle for space/momentum. If a particle's position in time is well-defined, its energy will highly uncertain and vice versa.

1.0.1 Dispersion in time

If the wave functions normally have an extension in time then every time-specific measurement should show additional dispersion in time.

Suppose we are measuring the time-of-arrival of a particle at a detector. Define the average time-of-arrival as:

$$\left\langle \tau^{(TOA)} \right\rangle \equiv \int\limits_{-\infty}^{\infty} d\tau \tau p^{(TOA)} (\tau) \tag{7}$$

with an associated uncertainty:

$$\left\langle \Delta\tau^{(TOA)} \right\rangle \equiv \sqrt{\int\limits_{-\infty}^{\infty} d\tau \tau^2 p^{(TOA)} (\tau) - \left\langle \tau^{(TOA)} \right\rangle^2} \tag{8}$$

The probability distribution for the particle will normally be spread out in space, so its arrival times will also be spread out, depending on the velocity of the particle and its dispersion in space.

But if it also has a dispersion in time, then part of the wave function will reach the detector – thanks to the fuzziness in time – a bit sooner and also a bit later than otherwise expected. There will be an additional dispersion in the time-of-arrival due to the dispersion in time.

As we will see (subsection 4.3), at non-relativistic speeds, the dispersion in the time-of-arrival is dominated by the dispersion in space, so this effect may be hard to pick out. At relativistic speeds, the contributions of the space and time dispersions can be comparable.

1.0.2 Uncertainty principle for time and energy

Of particular significance for this work are differences in the treatment of the uncertainty principle for time/energy as opposed to that for space/momentum.

In the early days of quantum mechanics, these were treated on same basis. See for instance the discussions between Bohr and Einstein of the famous clock-in-a-box experiment Schilpp and Bohr [1949] or the comments of Heisenberg in Heisenberg [1930].

In later work this symmetry was lost. As Busch Busch [2001] puts it "... different types of time energy uncertainty can indeed be deduced in specific contexts, but ... there is no unique universal relation that could stand on equal footing with the position-momentum uncertainty relation." See also Pauli, Dirac, and Muga Pauli [1980], Dirac [1958], Muga et al. [2002, 2008].

There does not appear to be any experimental test of this or observational evidence for it; it is merely the way the field has developed.

That is not to say that there are not uncertainties with respect to time, but they are side effects of other uncertainties in quantum mechanics. For instance, if a particle is moving to the right, spread out in space, and going towards a detector at a fixed position, its time-of-arrival will have a dispersion in time. But this is a side-effect of the dispersion in space.

Now consider a particle going through a narrow slit in time, for instance a camera shutter. Its wave function will be clipped in time. If the wave function is not extended in time, then the wave function will merely be clipped: the resulting dispersion in time at detector will be *reduced*.

But if the wave function is extended in time and the Heisenberg uncertainty principle applies in time/energy on the same basis as with space/momentum, then a very fast camera shutter will give a small uncertainty in time at the gate:

$$\Delta t \to 0 \tag{9}$$

causing the uncertainty in energy to become arbitrarily great:

$$\Delta E \geq \frac{1}{\Delta t} \Rightarrow \Delta E \to \infty \tag{10}$$

which will in turn cause the wave function to be diffracted, to fan out in time, and the dispersion in time-of-arrival to become arbitrarily great.

1.0.3   A necessary hypothesis

This question does not appear to have been attacked directly. As noted, the assumption that the wave function is not extended in time seems to have crept into the literature of its own, without experimental test or observational evidence.

To make an experimental test of this question we have to develop predictions for both branches:

1. Assume the wave function is not extended in time. Make predictions about time-of-arrival and the like.

2. Assume the wave function is extended in time. Make equivalent predictions.

3. Compare.

We have to develop both branches in a way that makes the comparison straightforward.

Further, to make the results falsifiable we have to develop the extended-in-time branch in a way that is clearly correct. A null result should show that the wave function is *not* extended in time.

These requirements motivate much of what follows.

1.0.4 Literature

The literature for special relativity and for quantum mechanics is vast. Our focus is on the critical intersection of the two. References of particular interest here include:

- Feynman's original papers: Feynman [1949a,b].

- Block universe picture, Barbour, Price: Barbour [2000], Price [1996].

- Reviews of the role of time, Schulman, Zeh, Muga, Callender: Schulman [1997], Zeh [2001], Muga et al. [2002, 2008], Callender [2017].

- Cramer's transactional interpretation:Cramer [1986, 1988], Kastner [2013], Cramer [2016].

- The time symmetric quantum mechanics of Aharonov and Reznik: Aharonov and Rohrlich [2005], Reznik and Aharonov [1995].

- The relativistic dynamics program of Horwitz, Land, and others: Land and Horwitz [1996], Gill et al. [2010], Horwitz [2015].

1.1 Order of magnitude estimate

Has this hypothesis has already been falsified? Quantum mechanics has been tested with extraordinary precision. Should associated effects have been seen already, even if not looked for?

Consider the atomic scale given by the Bohr radius $5.3 \ 10^{-11}m$. We take this as an estimate of the uncertainty in space.

We assume the maximum symmetry possible between time and space. We therefore infer that the uncertainty in time should be of order the uncertainty in space (in units where $c = 1$).

Dividing the Bohr radius by the speed of light we get the Bohr radius in time $a_0 = .177 \ 10^{-18}s$, or less than an attosecond. $.177as$ is therefore our starting estimate of the uncertainty in time.

Therefore from strictly dimensional and symmetry arguments, the effects will be small, of order attoseconds. This is sufficient to explain why such effects have not been seen.

At the same time, the time scales we can look at experimentally are now getting down to the attosecond range. A recent paper by Ossiander et al Ossiander et al. [2016] reports results at the sub-attosecond level.

Therefore if we can provide the experimentalists with a sufficiently well-defined target, the hypothesis should be falsifiable in practice.

1.2 Plan of attack

Look, I don't care what your theory of time is. Just give me something I can prove wrong. – experimentalist at the 2009 Feynman Festival in Olomouc

1.2.1 Primary objective is falsifiability

It is not enough to extend quantum mechanics to include time. It is necessary to do so in a way that can be proved wrong. The approach has to be so strongly and clearly constrained that if it is proved wrong, the whole project of extending quantum mechanics to include time is falsified.

Our requirements therefore are that we have:

1. the most complete possible equivalence in the treatment of time and space – manifest covariance at every point at a minimum,

2. consistency with existing experimental and observation results,

3. and consistency between the single particle and multiple particle domains.

These requirements leave us with no free parameters. And having no free parameters means in turn that our hypothesis is falsifiable in principle.

To get to falsifiable in practice, we will look for the simplest cases that make a direct comparison possible. We will also look at points of principle that that need to be addressed.

We will use the acronym SQM for standard quantum mechanics. We will use the acronym TQM for temporal quantum mechanics. By TQM we mean SQM with time treated on the system basis as space: time just as much an observable as the three space dimensions.

We do *not* mean by "temporal quantum mechanics" that time itself comes in small chunks or quanta! For instance, there has been speculation that time is granular at the scale of the Planck time: $t_{Planck} \equiv \sqrt{\frac{\hbar G}{c^5}} \approx 5.39116x10^{-44}s = 5.39116x10^{-26}as$. Perhaps it is, perhaps it isn't. But as this is 26 orders of magnitude smaller than the times we are considering here, it is reasonable for us to take time as continuous. And since space is treated by SQM as continuous, and since the defining assumption of TQM is the maximum symmetry between time and space, we are required to take time in TQM continuous.

1.2.2 Single particle case

In the single particle case we will:

1. Generalize path integrals to include time as an observable.

2. Derive the corresponding Schrödinger equation as the short time limit of the path integral.

3. Develop the free solutions. We will estimate the initial wave function, let it evolve in time, and detect it. We will compute the dispersions of time-of-arrival measurements in SQM and in TQM. In general the differences are real but small.

4. Develop the semi-classical approximation for TQM. We will show that TQM is to SQM (with respect to time) as SQM is to classical mechanics.

5. Analyze the single and double slit experiments. The single slit in time experiment provides the decisive test of temporal quantum mechanics. In SQM, the narrower the slit, the *less* the dispersion in subsequent time-of-arrival measurements. In TQM, the narrower the slit, the *greater* the dispersion in subsequent time-of-arrival measurements. *In principle, the difference may be made arbitrarily great.*

1.2.3   Multiple particle case

We will then extend TQM to include the multiple particle case, i.e. field theory. We will show, using a toy model, that:

1. We can extend the usual path integral approach to include time as an observable. The basis functions in Fock space extend in a natural way from three to four dimensions, the Lagrangian is unchanged, and the usual Feynman diagram expansions appear.

2. For each Feynman diagram in SQM we can compute the TQM equivalent. Therefore any problem that can be solved using Feynman diagrams in SQM can be solved in TQM.

3. The usual ultraviolet divergences do not appear (the combination of dispersion in time and entanglement in time contain them).

4. And that there are a large number of additional experimental effects to be seen, including:

    (a) wave functions anti-symmetric in time,
    (b) correlations in time (Bell's theorem in time),
    (c) and forces of anticipation and regret.

1.2.4   Overall conclusions

With this done, we will argue in the discussion:

1. that TQM is not ruled out a priori.

2. that TQM is falsifiable. And given experimental work like Ossiander's, probably with current technology.

3. that TQM is a source of interesting experiments. Every foundational experiment in SQM has an "in time" variant.

4. that TQM is a potential starting point for attacks on the quantum gravity problem, since TQM is manifestly covariant and untroubled by the ultraviolet divergences.

5. that as TQM is a straight-forward extrapolation of quantum mechanics and special relativity, experiments that falsify TQM are likely to require modification of our understanding of either quantum mechanics or special relativity or both. Something will have to break. (We suspect our Olomouc experimentalist will not much care which, so long as he gets to do the breaking.)

2 Path integrals

2.1 Overview

To extend quantum mechanics to include time we will take as our starting point Feynman's path integral approach to quantum mechanics Feynman et al. [2010], Schulman [1981], Rivers [1987], Swanson [1992], Khandekar et al. [1993], Kashiwa et al. [1997], Grosche and Steiner [1998], Zinn-Justin [2005], Kleinert [2009], Zee [2010].

With the path integral approach, the only change we will need to make is to generalize the paths from varying in three dimensions to varying in four.

To make clear what this means, consider the case of Alice walking her dog, say from her front door to Bob's.

Alice will take the shortest (classical) path from door to door.

But her dog will dart from side to side, now investigating a mailbox to the left, now checking out a lamppost to the right. In fact, as a quantum dog he will investigate all such paths simultaneously. While he will start at the same time and place as Alice, and finish at the same time and place as Alice, in between he will travel simultaneously along all possible paths.

But – in SQM – only along paths in space. At each tick of Alice's digital watch, her dog will be found off to the left or right, jumping up or digging down, further along the path to Bob's, or holding back for an important investigation.

But in TQM, the quantum dog can – and therefore will – advance into the future and drop back into the past. So that tick by tick of Alice's watch, her dog's paths will have to tracked in four dimensions rather than three.

This is harder to visualize, being out of our normal experience. So we develop the analysis a bit formally, letting math take the place of an as yet undeveloped intuition.

Path integrals, as the name suggests, are done by summing over all paths from starting point to finish, weighting each path by the integral of the Lagrangian (the action) along it:

$$\psi_T\left(x_T\right) = \int \mathcal{D}x_\tau \exp\left(\imath \int\limits_0^T d\tau L\left[x, \dot{x}\right]\right) \psi_0\left(x_0\right) \tag{11}$$

Piece by piece:

1. $\psi_0\left(x_0\right)$ is the initial wave function. We will be breaking these down into sums over Gaussian test functions using Morlet wavelet analysis.

2. τ is the clock time as given by Alice's digital watch. We will break up the paths into the bits from one clock tick to the next.

3. $\mathcal{D}x_\tau$ represents the paths. Each path is defined by its coordinates at each clock tick. In SQM, these are the values of x_τ, y_τ, z_τ at each clock tick. In TQM these are values of $t_\tau, x_\tau, y_\tau, z_\tau$ at each clock tick.

4. $L[x, \dot{x}]$ is a suitable Lagrangian. We will be using one that works equally well for both SQM and TQM.

5. $\imath \int\limits_0^T d\tau L[x, \dot{x}]$ is the action, the integral over the Lagrangian taken path by path.

6. And $\psi_T(x_T)$ is the final wave function, the amplitude for the dog to arrive at Bob's door step.

We will look at:

1. What do we mean by τ the clock time?

2. What do we mean by the coordinate time t in t, x, y, z?

3. How do we define the initial wave function in a way that does not potentially bias the outcome?

4. What Lagrangian shall we use?

5. How do we get the sums to converge?

6. Having gotten the sums to converge, how do we normalize them?

7. And what do all the pieces look like when we put them back together?

2.2   Laboratory time

In classical mechanics when we look at the action, at the integral of the Lagrangian over time:

$$\int\limits_0^T d\tau \mathcal{L}[x, \dot{x}] \tag{12}$$

we are free to take the parameter τ as any monotonically increasing variable. We will get the same classical equations of motion in any case.

A typical choice is to select τ as the proper time of the particle in question. However this makes it difficult to extend the work to the multiple particle case, where there are many particles and therefore many proper times at play.

Here we choose to use the time as shown on a laboratory clock. We take the term laboratory time from Busch Busch [2001]. We will use the terms clock time and laboratory time interchangeably.

It is useful to visualize this clock as a metronome, breaking up the clock time into a series of N ticks each of length ϵ. If $\tau = 0$ at the source, and $\tau = T$ at the detector, we have:

$$\epsilon \equiv \frac{T}{N} \tag{13}$$

We will take the limit as $N \to \infty$ as the final step in the calculation.

2.3 Coordinate time

We visualize a four dimensional coordinate system coordinates t, x, y, z. We will refer to t as coordinate time by analogy with the three coordinate space dimensions: coordinate x, coordinate y, and coordinate z.

Paths are defined with reference to this coordinate system. If the time by Alice's watch is τ, then each path π will have a location at τ given by:

$$\pi_\tau (t, x, y, z) \tag{14}$$

It may help to think of the coordinates as laid out on a piece of four dimensional graph paper. At a specific clock tick n, a specific path π will be represented by a dot on a specific vertex on the four dimensional graph paper. If we want to see the progress of the path with respect to clock time, we can flip the series of pieces of graph paper like one of those old time flip movies.

If our graph paper has M grid lines in each direction, the number of vertices on a page is M^4, and the number of paths total is M^{4N}. Each different sequence of grid points counts as a distinct path.

The path integral measure $\mathcal{D}x$ is usually defined by assigning a weight of one to each distinct path, and then taking the limit as the spacing goes to zero.

Since coordinate time is on the same footing as the three space coordinates there is a corresponding energy operator:

$$p_x \equiv -\imath \frac{\partial}{\partial x} \Rightarrow E \equiv \imath \frac{\partial}{\partial t} \tag{15}$$

We will refer to this as coordinate energy. It is not positive definite or bounded from below. Since p_x can be positive or negative, by our controlling requirement of covariance E can be positive or negative.

We discuss the relationship between clock time and coordinate time in detail in G.

2.4 Initial wave function

We need a starting set of wave functions ψ_0 at clock time $\tau = 0$. We will need wave functions that extend in both coordinate time and space. The usual choices would be δ functions or plane waves.

In coordinate time these might be:

$$\delta(t - t_0) \tag{16}$$

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$$e^{-\imath E(t-t_0)} \tag{17}$$

Or in space:

$$\delta(x - x) \tag{18}$$

$$e^{\imath p(x-x_0)} \tag{19}$$

But neither δ functions nor plane waves are physical. Their use creates a risk of artifacts.

More physical would be Gaussian test functions, for instance in coordinate time:

$$\varphi(t) \equiv \sqrt[4]{\frac{1}{\pi\sigma_t^2}} e^{-\imath E(t-t_0) - \frac{(t-t_0)^2}{2\sigma_t^2}} \tag{20}$$

Or in space:

$$\varphi(x) \equiv \sqrt[4]{\frac{1}{\pi\sigma_x^2}} e^{\imath p(x-x_0) - \frac{(x-x_0)^2}{2\sigma_x^2}} \tag{21}$$

But while Gaussian test functions are physically reasonable they are not completely general.

We can achieve both generality and physical reasonableness by using a basis of Morlet wavelets Morlet et al. [1982], Chui [1992], Meyer and Salinger [1992], Kaiser [1994], Berg [1999], Bratteli and Jørgensen [2002], Addison [2002], Visser [2003], Antoine et al. [2004], Ashmead [2012].
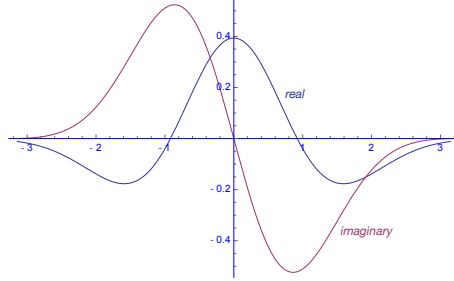
Figure 2: Mother Morlet wavelet

Morlet wavelets are derived by starting with a "mother" wavelet

$$\phi^{(mother)}(t) \equiv \left(e^{-\imath t} - \frac{1}{\sqrt{e}}\right) e^{\left(-\frac{t^2}{2}\right)} \tag{22}$$

and scaling and displacing it with the replacement $t \rightarrow \frac{t-l}{s}$

$$\phi_{sl}(t) = \frac{1}{\sqrt{|s|}} \left(e^{-\imath \left(\frac{t-l}{s} \right)} - \frac{1}{\sqrt{e}} \right) e^{-\frac{1}{2} \left(\frac{t-l}{s} \right)^2} \tag{23}$$

Any normalizable function f can be broken up into wavelet components $\hat{f}$ using

$$\hat{f}_{sl} = \int\limits_{-\infty}^{\infty} dt \phi_{sl}^*(t) f(t) \tag{24}$$

And then recovered using the inverse Morlet wavelet transform:

$$f(t) = \frac{1}{C} \int\limits_{-\infty}^{\infty} \frac{dsdl}{s^2} \phi_{sl}(t) \hat{f}_{sl} \tag{25}$$

The value of C is worked out in Ashmead [2012].

Therefore we can write any physically reasonable wave function in time as:

$$\psi(t) = \frac{1}{C} \int \frac{dsdl}{s^2} \hat{\psi}_{sl} \phi_{sl}(t) \tag{26}$$

And include space by using products of Morlet wavelets:

$$\psi(t,x) = \frac{1}{C^2} \int \frac{ds_t dl_t}{s_t^2} \frac{ds_x dl_x}{s_x^2} \hat{\psi}_{s_t l_t s_x l_x} \phi_{s_t l_t}(x) \phi_{s_x l_x}(x) \tag{27}$$

Clearly it would be cumbersome to track four dimensional Morlet wavelets at every step.

Fortunately we do not need to actually perform the Morlet wavelet analyses: we merely need the ability to do so. As each Morlet wavelet may be written as a sum of a pair of Gaussians, Morlet wavelet analysis lets us write any physically reasonable wave function as a sum over Gaussians. Provided we are dealing only with linear operations – the case throughout here – we can work directly with Gaussian test functions. By Morlet wavelet analysis the results will then be valid for any physically reasonable wave functions.

2.5 Lagrangian

To sum over the paths – to construct the path integral – we will need to weight each path by the exponential of the action, where the action is defined as the integral of the Lagrangian over the laboratory time:

$$e^{\imath \int\limits_0^T d\tau L(x^\mu, \dot{x}^\mu)} \tag{28}$$

We require a Lagrangian which:

1. Is manifestly covariant,

2. Produces the correct classical equations of motion,

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3. And gives the correct Schrödinger equation.

We would further prefer a Lagrangian which is the same for both SQM and TQM. This will let us argue that we are treating SQM and TQM with the most complete possible equality.

Somewhat surprisingly such a Lagrangian exists. In Goldstein's well-known text on classical mechanics Goldstein [1980] we find:

$$L\left(x^{\mu}, \dot{x}^{\mu}\right) = -\frac{1}{2} m \dot{x}^{\mu} \dot{x}_{\mu} - q \dot{x}^{\mu} A_{\mu}\left(x\right) \tag{29}$$

The potentials are not themselves functions of the laboratory time τ.

This Lagrangian is unusual in that it uses four independent variables (the usual three space coordinates plus a time variable) but still gives the familiar classical equations of motion (see B).

This Lagrangian therefore provides a natural bridge from a three to a four dimensional picture.

The classical equations of motion are still produced if we add a dimensionless scale a and an additive constant b to the Lagrangian:

$$-\frac{1}{2} a m \dot{x}^{\mu} \dot{x}_{\mu} - a q \dot{x}^{\mu} A_{\mu}\left(x\right) - a b \frac{m}{2} \tag{30}$$

The Lagrangian is therefore only determined up to a and b. The requirement that we match the SQM results will fix a and b (subsection 3.3).

2.6 Convergence

How do we get the sums to converge without breaking covariance?

We compute the path integral for the kernel by slicing the clock time into an infinite number of intervals and integrating over each:

$$K_{BA} = \lim_{N \to \infty} C_N \int \prod_{n=1}^{n=N} dt_n d\vec{x}_n e^{\imath \varepsilon \sum_{j=1}^{N+1} L_j} \tag{31}$$

with C_N an appropriate normalization factor.

Consider the discrete form of the Lagrangian. We use a tilde to mark the coordinate time part and an overbar to mark the space part:

$$L_j \equiv \tilde{L}_j^t + \bar{L}_j^{\vec{x}} + L_j^m \tag{32}$$

$$\tilde{L}_j^t \equiv -a \frac{m}{2}\left(\frac{t_j - t_{j-1}}{\varepsilon}\right)^2 - q a \frac{t_j - t_{j-1}}{\varepsilon} \frac{\Phi\left(x_j\right) + \Phi\left(x_{j-1}\right)}{2} \tag{33}$$

$$\bar{L}_j^{\vec{x}} \equiv a \frac{m}{2}\left(\frac{\vec{x}_j - \vec{x}_{j-1}}{\varepsilon}\right)^2 + q a \frac{\vec{x}_j - \vec{x}_{j-1}}{\varepsilon} \cdot \frac{\vec{A}\left(x_j\right) + \vec{A}\left(x_{j-1}\right)}{2} \tag{34}$$

$$L_j^m \equiv -a b \frac{m}{2} \tag{35}$$

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We are using the mid-point rule, averaging the scalar and the vector potentials over the start and end points of the step, by analogy with the rule for three dimensions (Schulman, Grosche and Steiner Schulman [1981], Grosche and Steiner [1998]).

Now look at a single step for the free case, vector potential A_μ zero:

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} dt_j d\vec{x} e^{-\imath a m \frac{(t_j - t_{j-1})^2}{2\varepsilon} + \imath a m \frac{(\vec{x}_j - \vec{x}_{j-1})^2}{2\varepsilon}} \tag{36}$$

The formal tricks normally used to ensure convergence do not work here (e.g. Kashiwa or Zinn-Justin Kashiwa et al. [1997], Zinn-Justin [2005]). Perhaps the most popular of these is the use of Wick rotation to shift to a Euclidean time:

$$\tau \to -\imath\tau \tag{37}$$

This causes integrals to converge rapidly going into the future, but makes the past inaccessible. For instance, factors of $\exp(-\imath\omega\tau)$ – which spring up everywhere in path integrals – converge going into the future, but blow up going into the past. If we are to treat time on the same footing as space – our central assumption – then past and future must be treated as symmetrically as left and right.

Another approach is to add a small convergence factor at a cleverly chosen spot in the arguments of the exponentials. But if we attach a convergence factor to t and x separately, we break manifest covariance. If we attach our convergence factor to both, the fact that the t and x parts enter with opposite sign means any convergence factor that works for one will fail for the other. We could try attaching one to the mass m, but this also fails. For instance if $a > 0$ and we subtract a small factor of $\imath\delta$ from the mass:

$$m \to m - \imath\delta \tag{38}$$

the t integral converges but the x integral diverges.

We recall the kernel has meaning only when applied to a specific physical wave function. If we break the incoming wave up into Morlet wavelets and then into Gaussian test functions, we see that each integral converges by inspection, the factor $e^{-\frac{1}{2}\left(\frac{t-l}{s}\right)^2}$ ensures this.

So for physically significant wave functions, there is no problem in the first place. Effectively we are taking seriously the point that the path integral kernel is a distribution, only meaningful with respect to specific wave functions.

2.7   Normalization

Now that we have our path integrals converging, we have to normalize them. If we start from the Schrödinger equation, the normalization is wired in. But in path integrals we are a bit at sea.

We will here deal with the free case, verifying the normalization is correct in the general case in D.

The normalization factor for N steps we will call C_N. The defining requirement is that, if the initial wave function is normalized to one, then with the inclusion of C_N, the final wave function will be normalized to one as well:

$$\int dt_0 d\vec{x}_0 \left| \psi_0(t_0, \vec{x}_0)^2 \right| = 1 \rightarrow \int dt_N d\vec{x}_N \left| \psi_N(t_N, \vec{x}_N)^2 \right| = 1 \qquad (39)$$

If C_N depends on the particular ψ_0, then we have failed.

We now compute the factor of C_N.

Normalization in time We start with the coordinate time dimension only. Consider a Gaussian test function centered on an initial position in coordinate time $\bar{t}_0$:

$$\tilde{\varphi}_0 \left(t_0 \right) \equiv \sqrt[4]{\frac{1}{\pi \sigma_t^2}} \, e^{-\imath E_0 t_0 - \frac{(t_0 - \bar{t}_0)^2}{2\sigma_t^2}} \qquad (40)$$

We write the kernel for the time part as:

$$\tilde{K}_\tau \left(t_N, t_0 \right) \sim \int \prod_{j=1}^{N} dt_j e^{-\imath a m \sum_{k=1}^{N} \frac{(t_k - t_{k-1})^2}{2\varepsilon}} \qquad (41)$$

The wave function after the initial integral over t_0 is:

$$\tilde{\varphi}_\varepsilon \left(t_1 \right) = \int dt_0 e^{-\imath \frac{am}{2\varepsilon}(t_1 - t_0)^2} \tilde{\varphi}_0 \left(t_0 \right) \qquad (42)$$

or:

$$\varphi_\varepsilon \left(t_1 \right) = \sqrt{\frac{2\pi\varepsilon}{\imath am}} \sqrt[4]{\frac{1}{\pi \sigma_0^2}} \sqrt{\frac{1}{f_\varepsilon^{(t)}}} \, e^{-\imath E_0 t_1 + \imath \frac{E_0^2}{2am}\varepsilon - \frac{1}{2\sigma_0^2 f_\varepsilon^{(t)}}\left(t_1 - \bar{t}_0 - \frac{E_0}{am}\varepsilon \right)^2} \qquad (43)$$

with the dispersion factor $f_\tau^{(t)} \equiv 1 - \imath \frac{\tau}{m\sigma_t^2}$.

The normalization requirement is:

$$1 = \int dt_1 \tilde{\varphi}_\varepsilon^* \left(t_1 \right) \tilde{\varphi}_\varepsilon \left(t_1 \right) \qquad (44)$$

The first step normalization is correct if we multiply the kernel by a factor of $\sqrt{\frac{\imath am}{2\pi\varepsilon}}$. Since this normalization factor does not depend on the laboratory time the overall normalization for N infinitesimal kernels is the product of N of these factors:

$$C_N \equiv \sqrt{\frac{\imath am}{2\pi\varepsilon}}^{N} \qquad (45)$$

Note also that the normalization does not depend on the specifics of the Gaussian test function (the values of E_0, σ_t^2, and $\bar{t}_0$) so it is valid for an arbitrary

sum of Gaussian test functions as well. And therefore, by Morlet wavelet decomposition, for an arbitrary wave function.

As noted, the phase is arbitrary. If we were working the other way, from Schrödinger equation to path integral, the phase would be determined by the Schrödinger equation itself. The specific phase choice we are making here has been chosen to help ensure the four dimensional Schrödinger equation is manifestly covariant, see below. We may think of the phase choice as a choice of gauge (see E).

Therefore the expression for the free kernel in coordinate time is (with $t'' \equiv t_N$, $t' \equiv t_0$):

$$\tilde{K}_\tau \left(t''; t'\right) = \int dt_1 dt_2 \dots dt_N \sqrt{\frac{\imath am}{2\pi\varepsilon}}^N e^{-\imath \sum_{j=1}^{N} \left(\frac{am}{2\varepsilon} (t_j - t_{j-1})^2 \right)} \tag{46}$$

Doing the integrals we get:

$$\tilde{K}_\tau \left(t''; t'\right) = \sqrt{\frac{\imath am}{2\pi\tau}} e^{-\imath am \frac{(t'' - t')^2}{2\tau}} \tag{47}$$

and free wave functions in coordinate time:

$$\tilde{\varphi}_\tau \left(t\right) = \sqrt[4]{\frac{1}{\pi\sigma_t^2}} \sqrt{\frac{1}{f_\tau^{(t)}}} e^{-\imath E_0 t - \frac{1}{2\sigma_t^2 f_\tau^{(t)}} \left(t - \bar{t}_0 - \frac{E_0}{am}\tau\right)^2 + \imath \frac{E_0^2}{2am}\tau} \tag{48}$$

Normalization in space   We redo the analysis for coordinate time for space. We use the correspondences:

$$t \to x, m \to -m, \bar{t}_0 \to \bar{x}_0, E_0 \to -p_0, \sigma_t^2 \to \sigma_x^2 \tag{49}$$

With these we can write down the equivalent set of results by inspection. Since we will need the results below, we do this explicitly. We get the initial Gaussian test function:

$$\bar{\varphi}_0 \left(x_0\right) = \sqrt[4]{\frac{1}{\pi\sigma_x^2}} e^{\imath p_0 x_0 - \frac{(x_0 - \bar{x}_0)^2}{2\sigma_x^2}} \tag{50}$$

free kernel:

$$\bar{K}_\tau \left(x''; x'\right) \sim \int dx_1 dx_2 \dots dx_N e^{\imath \sum_{j=1}^{N} \frac{am}{2\varepsilon} (x_j - x_{j-1})^2} \tag{51}$$

normalization constant:

$$\sqrt{\frac{2\imath\pi\varepsilon}{am}} \to \sqrt{\frac{2\pi\varepsilon}{\imath am}} \tag{52}$$

and normalized kernel:

20

$$\bar{K}_\tau\left(x''; x'\right) = \sqrt{-\frac{\imath a m}{2\pi\tau}}\, e^{\imath a m \frac{(x''-x')^2}{2\tau}} \tag{53}$$

The kernel matches the usual (non-relativistic) kernel Feynman and Hibbs [1965], Schulman [1981] if $a = 1$.

The wave function is:

$$\bar{\varphi}_\tau\left(x\right) = \sqrt[4]{\frac{1}{\pi\sigma_x^2}}\sqrt{\frac{1}{f_\tau^{(x)}}}\, e^{\imath p_0 x - \frac{1}{2\sigma_x^2 f_\tau^{(x)}}\left(x-\bar{x}_0-\frac{p_0}{am}\tau\right)^2 - \imath \frac{p_0^2}{2am}\tau} \tag{54}$$

with the definition of the dispersion factor $f_\tau^{(x)} = 1 + \imath\frac{\tau}{m\sigma_x^2}$ parallel to that for coordinate time (but with opposite sign for the imaginary part).

Normalization in time and space The full kernel is the product of the coordinate time kernel, the three space kernels, and the constant term $e^{-\imath\frac{abm}{2}\tau}$. We understand x to refer to coordinate time and all three space dimensions:

$$K_\tau\left(x''; x'\right) = -\imath\frac{a^2 m^2}{4\pi^2\tau^2}\, e^{-\frac{\imath a m}{2\tau}\left(x''-x'\right)^2 - \imath\frac{abm}{2}\tau} \tag{55}$$

We have done the analysis only for Gaussian test functions, but by Morlet wavelet decomposition it is completely general.

2.8   Formal expression

We now have the full path integral:

$$K_\tau\left(x''; x'\right) = \lim_{N\to\infty}\int \mathcal{D}x\, e^{\imath\sum_{j=1}^{N+1} L_j\epsilon} \tag{56}$$

with the measure:

$$\mathcal{D}x \equiv \left(-\imath\frac{m^2}{4\pi^2 a^2\varepsilon^2}\right)^{N+1}\prod_{n=1}^{n=N} d^4 x_n \tag{57}$$

and the discretized Lagrangian at each step:

$$L_j \equiv -am\frac{(x_j - x_{j-1})^2}{2\varepsilon^2} - aq\frac{x_j - x_{j-1}}{\epsilon}\frac{A\left(x_j\right) + A\left(x_{j-1}\right)}{2} - b\frac{m}{2} \tag{58}$$

We will show that $a = b = 1$ in the next section.

3 Schrödinger equation

3.1 Overview

The path integral and Schrödinger equation views are complementary. We need both to fully understand either.

We derive the Schrödinger equation from the path integral by taking the short time limit of the path integral form.

By comparing the result to the Klein-Gordon equation – and making a reasonable assumption about the long time evolution of the wave functions – we are able to fix the additive and scale constants in the Lagrangian.

The resulting equation looks like the Klein-Gordon equation over short times but shows some drift over longer times. We use some heuristic arguments to estimate the scale of the long term drift as of order picoseconds, a million times longer than the attosecond scale of the time dispersions we are primarily concerned with here. We will therefore be able to largely ignore this drift.

With a, b defined, we look at a further problem. We have done the derivation of path integral and Schrödinger equation forms from Alice's perspective. But what of Bob, jetting around like a fusion powered mosquito?

We resolve this conflict by arguing that we can find a natural rest frame that both can use. Starting with an argument of Weinberg's, we argue we can associate an energy-momentum tensor with spacetime. This means we can associate a local rest frame with spacetime. And this local rest frame can provide the neutral and agreed defining frame for TQM.

This will complete the formal development of TQM.

3.2 Derivation of the Schrödinger equation

Normally the path integral expression is derived from the Schrödinger equation. But because for us the path integral provides the defining formulation we need to run the analysis in the "wrong" direction.

Our starting point is a derivation of the path integral from the Schrödinger equation by Schulman Schulman [1981]. We run his derivation in reverse and with one extra dimension.

We start with the discrete form of the path integral. We consider a single step of length ϵ, taking $\epsilon \to 0$ at the end. Because of this, only terms first order in ϵ are needed.

Following Schulman, we define the coordinate difference:

$$\xi \equiv x_j - x_{j+1} \tag{59}$$

We rewrite the functions of x_j as functions of ξ and x_{j+1}. We expand the vector potential:

$$A_\nu\left(x_j\right) = A_\nu\left(x_{j+1}\right) + \left(\xi^\mu \partial_\mu\right) A_\nu\left(x_{j+1}\right) + \ldots \tag{60}$$

and the wave function:

$$\psi_\tau\left(x_j\right) = \psi_\tau\left(x_{j+1}\right) + \left(\xi^\mu \partial_\mu\right)\psi_\tau\left(x_{j+1}\right) + \frac{1}{2}\xi^\mu \xi^\nu \partial_\mu \partial_\nu \psi_\tau\left(x_{j+1}\right) + \dots \quad (61)$$

giving:

$$
\begin{aligned}
\psi_{\tau+\varepsilon}\left(x_{j+1}\right) = &\sqrt{\frac{\imath am}{2\pi\varepsilon}}\sqrt{-\frac{\imath am}{2\pi\varepsilon}}^3 \int d^4\xi\, e^{-\frac{\imath am\xi^2}{2\varepsilon}-\imath ab\frac{m}{2}\varepsilon} \\
&\times e^{\imath aq\xi^\nu\left(A_\nu(x_{j+1})+\frac{1}{2}\left(\xi^\mu\partial_\mu\right)A_\nu(x_{j+1})+\dots\right)} \\
&\times \left(\psi_\tau\left(x_{j+1}\right) + \left(\xi^\mu\partial_\mu\right)\psi_\tau\left(x_{j+1}\right) + \frac{1}{2}\xi^\mu\xi^\nu\partial_\mu\partial_\nu\psi_\tau\left(x_{j+1}\right) + \dots\right)
\end{aligned} \quad (62)
$$

We now expand in powers of $\xi \sim \sqrt{\epsilon}$. We do not need more than the second power:

$$
\begin{aligned}
\psi_{\tau+\varepsilon}\left(x_{j+1}\right) = &\sqrt{\frac{\imath am}{2\pi\varepsilon}}\sqrt{-\frac{\imath am}{2\pi\varepsilon}}^3 \int d^4\xi\, e^{-\frac{\imath am\xi^2}{2\varepsilon}} \\
&\times \left(1 + \imath aq\xi^\nu A_\nu\left(x_{j+1}\right) + \frac{\imath aq}{2}\xi^\nu\xi^\mu\partial_\mu A_\nu\left(x_{j+1}\right) - \frac{a^2q^2}{2}\xi^\mu A_\mu\left(x_{j+1}\right)\xi^\nu A_\nu\left(x_{j+1}\right) - \imath\frac{abm\varepsilon}{2}\right) \\
&\times \left(\psi_\tau\left(x_{j+1}\right) + \left(\xi^\mu\partial_\mu\right)\psi_\tau\left(x_{j+1}\right) + \frac{1}{2}\xi^\mu\xi^\nu\partial_\mu\partial_\nu\psi_\tau\left(x_{j+1}\right) + \dots\right)
\end{aligned} \quad (63)
$$

The term zeroth order in ξ gives:

$$\sqrt{\frac{\imath am}{2\pi\varepsilon}}\sqrt{-\frac{\imath am}{2\pi\varepsilon}}^3 \int d\xi^4 e^{-\frac{\imath am\xi^2}{2\varepsilon}} = \sqrt{\frac{\imath am}{2\pi\varepsilon}}\sqrt{-\frac{\imath am}{2\pi\varepsilon}}^3 \sqrt{\frac{2\pi\varepsilon}{\imath am}}\sqrt{\frac{2\pi\varepsilon}{-\imath am}}^3 = 1 \quad (64)$$

This is not surprising; the normalization above was chosen to do this.

Terms linear in ξ give zero when integrated. The terms second order in ξ (first in ϵ) are:

$$\left(\begin{array}{l} -\imath\frac{abm\varepsilon}{2} + \imath aq\left(\xi^\nu A_\nu\right)\left(\xi^\mu\partial_\mu\right) + \frac{\imath aq}{2}\xi^\nu\xi^\mu\left(\partial_\mu A_\nu\right) \\ -\frac{a^2q^2}{2}\left(\xi^\mu A_\mu\right)\left(\xi^\nu A_\nu\right) + \frac{1}{2}\xi^\mu\xi^\nu\partial_\mu\partial_\nu \end{array}\right)\psi_\tau \quad (65)$$

Integrals over off-diagonal powers of order ξ^2 give zero. Integrals over diagonal ξ^2 terms give:

$$
\begin{aligned}
\sqrt{\frac{\imath am}{2\pi\varepsilon}}\int d\xi_0 e^{-\frac{\imath am\xi_0^2}{2\varepsilon}}\xi_0^2 &= \frac{\varepsilon}{\imath am} \\
\sqrt{-\frac{\imath am}{2\pi\varepsilon}}\int d\xi_i e^{\frac{\imath am\xi_i^2}{2\varepsilon}}\xi_i^2 &= -\frac{\varepsilon}{\imath am}
\end{aligned} \quad (66)
$$

The expression for the wave function is therefore:

$$\psi_{\tau+\varepsilon} = \psi_\tau - \frac{\imath abm\varepsilon}{2}\psi_\tau + \frac{q\varepsilon}{m}\left(A^\mu\partial_\mu\right)\psi_\tau + \frac{q}{2m}\varepsilon\left(\partial^\mu A_\mu\right)\psi_\tau + \frac{\imath aq^2\varepsilon}{2m}A^2\psi_\tau - \frac{\imath\varepsilon}{2ma}\partial^2\psi_\tau \quad (67)$$

Taking the limit $\epsilon \to 0$ and multiplying by $\imath$, we get the Schrödinger equation for TQM:

$$\imath\frac{\partial\psi_\tau}{\partial\tau} = ab\frac{m}{2}\psi_\tau + \frac{\imath q}{m}\left(A^\mu\partial_\mu\right)\psi_\tau + \frac{\imath q}{2m}\left(\partial^\mu A_\mu\right)\psi_\tau - \frac{aq^2}{2m}A^2\psi_\tau + \frac{1}{2ma}\partial^2\psi_\tau \quad (68)$$

or:

$$\imath\frac{\partial\psi_\tau}{\partial\tau}\left(t,\vec{x}\right) = -\frac{1}{2ma}\left(\left(\imath\partial_\mu - aqA_\mu\left(t,\vec{x}\right)\right)\left(\imath\partial^\mu - aqA^\mu\left(t,\vec{x}\right)\right) - a^2bm^2\right)\psi_\tau\left(t,\vec{x}\right)$$
$$(69)$$

If we make the customary identifications $\imath\frac{\partial}{\partial t} \to E$, $-\imath\vec{\nabla} \to \hat{\vec{p}}$ or $\imath\partial_\mu \to p_\mu$ we have:

$$\imath\frac{\partial\psi_\tau}{\partial\tau} = -\frac{1}{2ma}\left(\left(p_\mu - aqA_\mu\right)\left(p^\mu - aqA^\mu\right) - a^2bm^2\right)\psi_\tau \quad (70)$$

3.3 Long, slow approximation

We can now fix the scale and additive constants by looking at the behavior of the Schrödinger equation over longer times.

In his development of quantum mechanics from a time-dependent perspective Tannor [2007], Tannor used a requirement of constructive interference in time to derive the Bohr condition for the allowed atomic orbitals. We use a similar approach here.

If we average over a sufficiently long period of time, the results will be dominated by the components with:

$$\imath\frac{\partial\psi_\tau\left(x\right)}{\partial\tau} = 0 \quad (71)$$

The argument here is not that the typical variation from the long, slow solution is small, but rather that over time interactions with the system in question will tend to be dominated by interactions with the stabler, slower moving components. Interactions with more rapidly varying components will tend to average to zero.

Accepting this, then the right side looks like the Klein-Gordon equation. To complete this identification, first look at the case with the vector potential A zero:

$$\left(\hat{p}^2 - a^2bm^2\right)\psi = 0 \to a^2b = 1 \quad (72)$$

Now when A is not zero we have:

$$\left(\left(\hat{p} - aqA\right)^2 - m^2\right)\psi = 0 \to a = 1 \to b = 1 \quad (73)$$

We will refer to this as the "long, slow approximation".

In the free case, the long, slow approximation picks out the on-shell components:

$$\left(\hat{p}^2 - m^2\right)\psi = 0 \tag{74}$$

And more generally the solutions of the Klein-Gordon equation with the minimal substitution $p \to p - qA$:

$$\left((\hat{p} - qA)^2 - m^2\right)\psi = 0 \tag{75}$$

The two constants are now fixed. For the record, the full Schrödinger equation is:

$$\imath\frac{\partial\psi_\tau}{\partial\tau}(t,\vec{x}) = -\frac{1}{2m}\left((\imath\partial_\mu - qA_\mu(t,\vec{x}))(\imath\partial^\mu - qA^\mu(t,\vec{x})) - m^2\right)\psi_\tau(t,\vec{x}) \tag{76}$$

and in momentum space:

$$\imath\frac{\partial\psi_\tau}{\partial\tau} = -\frac{1}{2m}\left((p_\mu - qA_\mu)(p^\mu - qA^\mu) - m^2\right)\psi_\tau \tag{77}$$

And the free Schrödinger equation is:

$$2m\imath\frac{\partial\psi_\tau}{\partial\tau}(t,\vec{x}) = \left(\partial_\mu\partial^\mu + m^2\right)\psi_\tau(t,\vec{x}) \tag{78}$$

and in momentum space:

$$2m\imath\frac{\partial\psi_\tau}{\partial\tau} = -\left(p_\mu p^\mu - m^2\right)\psi_\tau(t,\vec{x}) \tag{79}$$

3.4 How long and how slow?

We have argued that we can fix the scaling and additive constants by looking at the behavior of the Schrödinger equation over long times. What do we mean by long times?

To see the relevant scale, we estimate the clock frequency f:

$$f \sim -\frac{E^2 - \vec{p}^2 - m^2}{2m} \tag{80}$$

We will argue (subsection 4.1.2) that in the non-relativistic case E is of order mass plus kinetic energy:

$$E \sim m + \frac{\vec{p}^2}{2m} \tag{81}$$

So we have:

$$E^2 - \vec{p}^2 - m^2 \sim \left(m + \frac{\vec{p}^2}{2m}\right)^2 - \vec{p}^2 - m^2 = \left(\frac{\vec{p}^2}{2m}\right)^2 \tag{82}$$

This is just the kinetic energy, squared. In an atom the kinetic energy is of order the binding energy:

$$\frac{\vec{p}^2}{2m} \sim eV \tag{83}$$

So the numerator is of order eV squared. But the denominator is of order MeV. So we can estimate the clock frequency f as:

$$f \sim \frac{eV^2}{MeV} \sim 10^{-6} eV \tag{84}$$

Energies of millionths of an electron volt $10^{-6} eV$ correspond to times of order millions of attoseconds $10^6 as$ or picoseconds, a million times longer than the natural time scale of the effects we are looking at. So the long, slow approximation is reasonable.

3.5 Observer independent choice of frame

If we did the above derivation for Bob rather than Alice, we would see his clock time $\tau \to \gamma\tau$ where $\gamma \equiv 1/\sqrt{1 - v^2}$ where v is his velocity relative to hers.

We therefore have one free parameter left to fix before we can declare our analysis free of free parameters.

We could argue that if Bob is not going that quickly, that the errors created by not fixing the frame this will introduce only small corrections, which will be of second order and therefore not relevant for falsifiability.

However establishing frame independence is interesting as a point of principle. This may be done in a natural way by making use of an observation from Weinberg Weinberg [1972]. Per Weinberg, we may treat the Einstein field equation for general relativity as representing conservation of energy-momentum when exchanges of energy momentum with spacetime are included.

Consider the Einstein field equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{85}$$

Rewrite as:

$$(G^{\mu\nu} + 8\pi G T^{\mu\nu})_{;\nu} = 0 \tag{86}$$

We may use this to associate an energy momentum tensor ($t_{\mu\nu}$ in Weinberg's notation) with local space time. Define:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{87}$$

where $h_{\mu\nu}$ vanishes at infinity but is not assumed small. The part of the Ricci tensor linear in h is:

$$R_{\mu\nu}^{(1)} \equiv \frac{1}{2}\left(\frac{\partial^2 h_\lambda^\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_\mu^\lambda}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h_\nu^\lambda}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right) \tag{88}$$

The exact Einstein equations may be written as:

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda} = -8\pi G\left(T_{\mu\nu} + t_{\mu\nu}\right) \tag{89}$$

where $t_{\mu\nu}$ is defined by:

$$t_{\mu\nu} \equiv \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\lambda}_{\lambda} - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}_{\lambda}\right) \tag{90}$$

Weinberg argues we may interpret $t_{\mu\nu}$ as the energy-momentum of the gravitational field itself.

We change to a coordinate frame in which $t_{\mu\nu}$ is diagonalized. We will refer to this as the rest frame of the vacuum or the V frame. We can treat this V frame as the defining frame for the four dimensional Schrödinger equation. As the V frame is invariant (up to rotations in three-space) we now have an invariant definition of the four dimensional Schrödinger equation.

This is obviously going to be a free-falling frame. So Alice and Bob – if they are working in a terrestrial laboratory – will have to adjust their calculations to include a correction for the upwards force the laboratory floor exerts against them. If their colleague Vera is working in an orbiting laboratory, she will be able to calculate without correction.

4 Free solutions

We now have the Schrödinger equation. What do its free solutions look like?

We examine in turn the birth, life, and death of a free particle. The calculations are straightforward. But each stage will present problems specific to TQM.

4.1 Initial wave function

What do our wave functions look like at start?

We have a chicken and egg problem here. Any initial wave function had itself to come from somewhere. How can we estimate the initial wave functions without first knowing them?

We look specifically at the Klein-Gordon equation for a static potential, one with:

$$A_0 = q\Phi\left(\vec{x}\right), \vec{A} = 0 \tag{91}$$

This includes the free case as the special case when $\Phi = 0$.

In SQM the Klein-Gordon equation is:

$$\left(\left(\imath\partial_\tau - q\Phi\left(\vec{x}\right)\right)^2 - \nabla^2 - m^2\right)\bar{\psi}_\tau\left(x\right) = 0 \tag{92}$$

In TQM the equivalent is the Schrödinger equation:

$$\imath \frac{\partial}{\partial \tau} \psi_\tau \left(t, \vec{x} \right) = -\frac{1}{2m} \left(\left(\imath \partial_t - q\Phi \left(\vec{x} \right) \right)^2 - \nabla^2 - m^2 \right) \psi_\tau \left(t, \vec{x} \right) \tag{93}$$

The long, slow approximation picks out the solutions with:

$$\left(\left(\imath \partial_t - q\Phi \left(\vec{x} \right) \right)^2 - \nabla^2 - m^2 \right) \psi_\tau \left(t, \vec{x} \right) = 0 \tag{94}$$

We will assume we are already in possession of solutions to the SQM version of the problem. We can leverage the SQM solutions in two different ways to get the TQM solution:

1. *Separation of variables.* Each SQM solution induces a TQM solution, which is basically the SQM solution with a plane wave bolted on. This is technically correct, but unphysical.

2. *Maximum entropy.* We can use the long, slow approximation to estimate the mean and uncertainty of the coordinate energy. With these we can use the method of Lagrange multipliers to get the maximum entropy solution. Maximum entropy solutions tend to be robust: even if we are wrong about the details, the order of magnitude should be correct. This will provide our preferred starting point.

As a quick check on the sanity of all this, we will use the virial theorem to estimate the TQM version of the atomic wave functions. The width in time/energy of these matches the initial order of magnitude estimate we gave in the introduction.

4.1.1   Solution by separation of variables

We consider the Klein-Gordon equation for potentials constant in time. We assume the magnetic field is zero:

$$\left(\left(E - \bar{V} \left(\vec{x} \right) \right)^2 - \vec{p}^2 - m^2 \right) \varphi \left(t, \vec{x} \right) = 0 \tag{95}$$

This includes attractive potentials as well as scattering potentials, provided they are constant in time. Expanded, the equation is:

$$\left(-\frac{\partial^2}{\partial t^2} - 2\bar{V} \left(\vec{x} \right) \imath \frac{\partial}{\partial t} + \bar{V} \left(\vec{x} \right)^2 - \vec{p}^2 - m^2 \right) \varphi \left(t, \vec{x} \right) = 0 \tag{96}$$

We solve this using separation of variables, looking for solutions of the form:

$$\varphi \left(t, \vec{x} \right) = \tilde{\phi}_n \left(t \right) \bar{\varphi}_n \left(\vec{x} \right) \tag{97}$$

We assume we have a solution for the standard Klein-Gordon equation:

$$\left(\left(\bar{E}_n - \bar{V} \left(\vec{x} \right) \right)^2 - \vec{p}^2 - m^2 \right) \bar{\varphi}_n \left(\vec{x} \right) = 0 \tag{98}$$

Then the coordinate time part is:

$$\tilde{\phi}_n(t) = \frac{1}{\sqrt{2\pi}} e^{-\imath \bar{E}_n t} \tag{99}$$

Because the potential is constant in time each use of the operator E turns into a constant E_n via:

$$E \to \imath \frac{\partial}{\partial t} \to \bar{E}_n \tag{100}$$

We get immediately:

$$\left(\left(\bar{E}_n - \bar{V}(\vec{x}) \right)^2 - \vec{p}^2 - m^2 \right) \varphi_n(t, \vec{x}) = 0 \tag{101}$$

So every solution of the Klein-Gordon equation in SQM generates a corresponding solution in TQM.

We could accomplish the same thing, formally, by taking $\tau \to t$. Since we expect in general that $\langle t \rangle \approx \tau$ (discussed further in the next section) this will often a give reasonable first approximation.

However this is not entirely satisfactory. We have a solution which is "fuzzy" in space, but "crisp" in time. A more realistic, if more complex solution, would include off-shell components. Even if our wave function started out as a simple plane wave in time, internal decoherence would rapidly turn it into something a bit more cloud-like. While mathematically acceptable, our solution is not physically plausible.

4.1.2 Solution by maximum entropy

The long, slow approximation picks out a single solution. But in practice we expect there would be a great number of solutions, with the one given by the long slow approximation merely the most typical.

Such a sum will have an associated probability density function. We can get a reasonable first estimate of this by defining appropriate constraints and then using the method of Lagrange multipliers to pick out the distribution with maximum entropy.

From the probability density, we will infer the wave function.

We start with the free case, as representing the simplest case of a constant potential. We treat the bound case below. We assume we are given the SQM wave function in three dimensions $\bar{\phi}(\vec{x})$.

We take as the constraints the norm, the expectation of the energy, and the expectation of the energy squared:

$$\begin{aligned} \langle 1 \rangle &= 1 \\ \bar{E} \equiv \langle E \rangle &= \sqrt{m^2 + \langle \vec{p} \rangle^2} \\ \langle E^2 \rangle &= \langle m^2 + \vec{p}^2 \rangle = m^2 + \langle \vec{p}^2 \rangle \end{aligned} \tag{102}$$

The uncertainty in energy is defined as:

$$\Delta E \equiv \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \tag{103}$$

The expectations are defined as integrals over the probability density

$$\langle f \rangle \equiv \int d\vec{p}\,\vec{\rho}\,(\vec{p})\, f\,(\vec{p}) \tag{104}$$

The constraints imply $\Delta E = \Delta p$.

We will work with box normalized energy eigenfunctions:

$$\phi_n\,(t) \equiv \frac{1}{\sqrt{2T}} e^{-\imath E_0 n t} \tag{105}$$

$$E_0 \equiv \frac{\pi}{T} \tag{106}$$

where n is an integer running from negative infinity to positive and the eigenfunctions are confined to a box extending T seconds into the future and T seconds into the past, where T is much larger than any time of interest to us. (We use a similar approach in developing the four dimensional Fock space below, subsection 7.3.2).

A general wave function can be written as:

$$\psi\,(t) = \sum_{n=-\infty}^{\infty} c_n \phi_n\,(t) \tag{107}$$

The coefficients c only appear as the square:

$$\rho_n \equiv c_n^* c_n \tag{108}$$

Expressed in this language we have the constraints:

$$
\begin{aligned}
C_0 &\equiv \sum_{n=-\infty}^{\infty} \rho_n - 1 = 0 \\
C_1 &\equiv E_0 \sum_{n=-\infty}^{\infty} n \rho_n - \langle E \rangle = 0 \\
C_2 &\equiv E_0^2 \sum_{n=-\infty}^{\infty} n^2 \rho_n - \langle E^2 \rangle = 0
\end{aligned}
\tag{109}
$$

We would like to find the solution that maximizes the entropy:

$$S \equiv \sum_{n=-\infty}^{\infty} -\rho_n \ln\,(\rho_n) \tag{110}$$

We form the Lagrangian from the sum of the entropy and the constraints, with Lagrange multipliers λ_0, λ_1, and λ_2:

$$L \equiv S + \lambda_0 C_0 + \lambda_1 C_1 + \lambda_2 C_2 \tag{111}$$

To locate the configuration of maximum entropy, we take the derivative with respect to the ρ_n:

$$\frac{\partial L}{\partial \rho_n} = 0 \tag{112}$$

getting:

$$-\ln \rho_n - 1 + \lambda_0 + E_0 n \lambda_1 + E_0^2 n^2 \lambda_2 = 0 \tag{113}$$

Therefore the distribution of the ρ_n is given by an exponential with zeroth, first, and second powers of the energy:

$$\rho(E) \sim e^{-a - bE - cE^2} \tag{114}$$

The constraints force the constants:

$$\rho(E) = \frac{1}{\sqrt{2\pi \Delta E^2}} e^{-\frac{(E - \bar{E})^2}{2\Delta E^2}} \tag{115}$$

We therefore have the probability density in energy; now we wish to estimate the corresponding wave function in energy (or equivalently time). There are two distinct ways to do this. First for each value of the wave function in momentum, we can posit an associated δ function in energy:

$$\hat{\varphi}(E, p) \sim \delta(E - E_p) \, \hat{\bar{\varphi}}(p) \tag{116}$$

$$E_p \equiv \sqrt{m^2 + p^2} \tag{117}$$

Or we can write the total wave function as the direct product of a wave function in energy times the postulated wave function in momentum:

$$\hat{\varphi}(E, p) \sim \hat{\bar{\varphi}}(E) \, \hat{\bar{\varphi}}(p) \tag{118}$$

The first approach implies a detailed specification of the distribution of the wave function in energy. If the wave function in momentum has a complex shape, so too will the energy part. The second approach is significantly simpler and therefore more robust.

We start with a Gaussian test function in momentum:

$$\hat{\bar{\varphi}}_0(p) = \sqrt[4]{\frac{1}{\pi \sigma_p^2}} e^{-\imath(p - p_0)x_0 - \frac{(p - p_0)^2}{2\sigma_p^2}} \tag{119}$$

With a corresponding wave function in space:

$$\bar{\varphi}_0(x) = \sqrt[4]{\frac{1}{\pi \sigma_x^2}} e^{\imath p_0 x - \frac{(x - x_0)^2}{2\sigma_x^2}} \tag{120}$$

The wave function in momentum implies by the arguments above a wave function of the form:

$$\hat{\bar{\varphi}}_0(E) = \sqrt[4]{\frac{1}{\pi \sigma_E^2}} e^{\imath(E - E_0)t_0 - \frac{(E - E_0)^2}{2\sigma_E^2}} \tag{121}$$

$$\sigma_E^2 = \sigma_p^2 \tag{122}$$

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$$E_0 = \sqrt{m^2 + \bar{p}^2} \tag{123}$$

Taking the Fourier transform of the energy part we have:

$$\tilde{\varphi}_0(t) = \sqrt[4]{\frac{1}{\pi \sigma_t^2}} e^{-\imath E_0(t-t_0) - \frac{t^2}{2\sigma_t^2}} \tag{124}$$

$$\sigma_t^2 = \frac{1}{\sigma_E^2} \tag{125}$$

We set $t_0 = 0$ as the overall phase is already supplied by the space/momentum part.

The full wave functions are the products of the coordinate time and space (or energy and momentum) parts:

$$\varphi_0(t, x) = \tilde{\varphi}_0(t) \bar{\varphi}_0(x) \tag{126}$$

$$\hat{\varphi}_0(E, p) = \hat{\tilde{\varphi}}_0(E) \hat{\bar{\varphi}}_0(p) \tag{127}$$

4.1.3 Bound state wave functions

We extend this approach to estimate the dispersion of a bound wave function in time.

In the case of a Coulomb potential we can estimate Δp from the virial theorem:

$$\left\langle \frac{\bar{p}^2}{2m} \right\rangle = -\frac{1}{2} \langle V \rangle \tag{128}$$

which implies:

$$\left\langle \frac{\bar{p}^2}{2m} \right\rangle + \langle V \rangle = \bar{E}_n \rightarrow \left\langle \frac{\bar{p}^2}{2m} \right\rangle = -\bar{E}_n \tag{129}$$

Since the average momentum is zero, we have the estimate:

$$\Delta E = \sqrt{-2m\bar{E}_n} \tag{130}$$

Substituting the mass of the electron and the Rydberg constant, we have:

$$\Delta E = \sqrt{2 \cdot 13.6\text{eV} \cdot (0.511 \cdot 10^6)\,\text{eV}} = 3728 \text{ eV} \tag{131}$$

And the corresponding dispersion in coordinate time is:

$$\Delta t = \frac{\hbar}{\Delta E} = .1766\,as \tag{132}$$

This matches the order of magnitude estimate we started with (subsection 1.1). The numerical closeness is coincidental, but does increase confidence the order of magnitude is correct.

4.2 Evolution of the free wave function

Now that we know what our starting wave functions look like, how do they evolve over time?

For a first examination we work with the non-relativistic case; we will extend to the relativistic case below (section 7). We will first look at the SQM case, then look at the TQM case, and then compare the two.

4.2.1 Evolution in SQM

We first look at the familiar problem of the evolution of the non-relativistic wave function with respect to clock time. We work with two dimensions – t, x – since the extension to y, z is straightforward.

The non-relativistic Schrödinger equation is:

$$\imath \frac{\partial}{\partial \tau} \bar{\psi} = \frac{p^2}{2m} \bar{\psi} = -\frac{\partial_x^2}{2m} \bar{\psi} \tag{133}$$

At clock time zero we start with a Gaussian test function in momentum with average position x_0, average momentum p_0, and dispersion in momentum σ_p. To reduce clutter we are using p for p_x:

$$\hat{\bar{\varphi}}_0 (p) = \sqrt[4]{\frac{1}{\pi \sigma_p^2}} e^{-\imath p x_0 - \frac{(p-p_0)^2}{2\sigma_p^2}} \tag{134}$$

In momentum space the problem is trivial. The solution is:

$$\hat{\bar{\varphi}}_\tau (p) = \sqrt[4]{\frac{1}{\pi \sigma_p^2}} e^{-\imath p x_0 - \frac{(p-p_0)^2}{2\sigma_p^2} - \imath \frac{p^2}{2m} \tau} \tag{135}$$

In coordinate space we get:

$$\bar{\varphi}_\tau (x) = \sqrt[4]{\frac{1}{\pi \sigma_x^2}} \sqrt{\frac{1}{f_\tau^{(x)}}} e^{\imath p_0 x - \frac{1}{2\sigma_x^2 f_\tau^{(x)}} \left(x - \bar{x}_0 - \frac{p_0}{m} \tau\right)^2 - \imath \frac{p_0^2}{2m} \tau} \tag{136}$$

with:

$$\sigma_x = \frac{1}{\sigma_p} \tag{137}$$

$$f_\tau^{(x)} = 1 + \imath \frac{\tau}{m \sigma_x^2} \tag{138}$$

4.2.2 Evolution in TQM

In two dimensions the Schrödinger equation for TQM is:

$$\imath \frac{\partial \psi_\tau}{\partial \tau} (t, \vec{x}) = -\frac{E^2 - p^2 - m^2}{2m} \psi_\tau (t, \vec{x}) = \left(\frac{\partial_t^2}{2m} - \frac{\partial_x^2}{2m} + \frac{m}{2}\right) \psi_\tau (t, \vec{x}) \tag{139}$$

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This looks much like the non-relativistic case, but with one extra space dimension.

We start in energy momentum space. The momentum part is as above. We start with a Gaussian test function in energy, with average time at start t_0, average energy E_0, and dispersion in energy σ_E:

$$\hat{\tilde{\varphi}}_0\left(E\right) \equiv \sqrt[4]{\frac{1}{\pi\sigma_E^2}}\, e^{\imath Et_0 - \frac{(E-E_0)^2}{2\sigma_E^2}} \tag{140}$$

Again we merely push this forward in clock time:

$$\hat{\psi}_\tau\left(E,p\right) = \hat{\tilde{\varphi}}_0\left(E\right)\hat{\tilde{\varphi}}_0\left(p\right)\exp\left(-\imath\frac{E^2 - p^2 - m^2}{2m}\tau\right) \tag{141}$$

We divide up the pieces of the clock time part, assigning the $\frac{E^2}{2m}$ to the energy part, the $\frac{p^2}{2m}$ to the momentum part, and keeping the third part outside:

$$\hat{\psi}_\tau\left(E,p\right) = \hat{\tilde{\varphi}}_\tau\left(E\right)\hat{\tilde{\varphi}}_\tau\left(p\right)\exp\left(\imath\frac{m}{2}\tau\right) \tag{142}$$

Now the energy part works in parallel to the momentum part:

$$\hat{\tilde{\varphi}}_\tau\left(E\right) \equiv \sqrt[4]{\frac{1}{\pi\sigma_E^2}}\, e^{\imath Et_0 - \frac{(E-E_0)^2}{2\sigma_E^2} - \imath\frac{E_0^2}{2m}\tau} \tag{143}$$

And in coordinate space:

$$\tilde{\varphi}_\tau\left(t\right) = \sqrt[4]{\frac{1}{\pi\sigma_t^2}}\sqrt{\frac{1}{f_\tau^{(t)}}}\, e^{-\imath E_0 t + \imath\frac{E_0^2}{2m}\tau - \frac{1}{2\sigma_t^2 f_\tau^{(t)}}\left(t - t_0 - \frac{E_0}{m}\tau\right)^2} \tag{144}$$

with ancillary definitions:

$$\sigma_E = \frac{1}{\sigma_t} \tag{145}$$

$$f_\tau^{(t)} \equiv 1 - \imath\frac{\tau}{m\sigma_t^2} \tag{146}$$

and with the expectation for coordinate time:

$$\bar{t}_\tau = \bar{t}_0 + \frac{E_0}{m}\tau \tag{147}$$

implying a velocity for coordinate time with respect to laboratory time:

$$v_0 = \frac{E_0}{m} \tag{148}$$

We define as usual:

$$\gamma \equiv \frac{E}{m} \tag{149}$$

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And have in the non-relativistic case, $\gamma \approx 1$. So the expectation of the coordinate time advances at the traditional one-second-per-second rate relative to the clock time.

4.2.3 Comparison of TQM to SQM

The TQM and SQM approaches develop in close parallel, with one peculiarity.

We are relying on the long, slow approximation. Especially over short times, this means that the total wave function appears to be relatively static with respect to evolution in clock time:

$$\imath \frac{\partial}{\partial \tau} \psi \approx 0 \tag{150}$$

In momentum space:

$$f_p \equiv -\frac{E^2 - p^2 - m^2}{2m} \approx 0 \tag{151}$$

Real wave functions are not, of course, static with respect to clock time.

The resolution is that most of the clock time dependence is carried by the coordinate time:

$$\frac{d}{d\tau} \psi = \frac{\partial}{\partial \tau} \psi + \frac{dt}{d\tau} \frac{\partial}{\partial t} \psi \approx \frac{dt}{d\tau} \frac{\partial}{\partial t} \psi \tag{152}$$

And in the non-relativistic case we have:

$$\frac{dt}{d\tau} = \frac{E}{m} \approx 1 \tag{153}$$

So we get as a rough approximation:

$$\frac{d}{d\tau} \psi \approx \frac{\partial}{\partial t} \psi \tag{154}$$

We see that the expectation of the coordinate time is about equal to the clock time. While Alice's dog is always getting ahead of and behind her, on average his position is about equal to hers:

$$\langle t \rangle \approx \tau \tag{155}$$

4.3 Time of arrival measurements

So we know what our wave function looks at at start and how it evolves with time. To complete the analysis of the free case we look how it is detected.

We look specifically at the measurement of time-of-arrival. We assume we have a particle going left to right, starting at $x = 0$. We place a detector at position $x = L$. It records when it detects the particle. The metric we are primarily interested in is the dispersion in time-of-arrival at the detector.
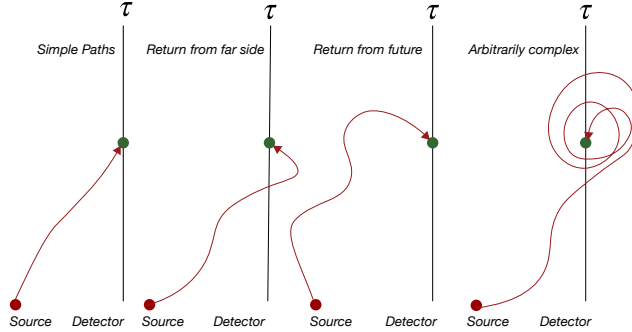
Figure 3: Complex paths near the detector

In SQM, if a detector located at position X registers a hit by a particle we take the particle's position in space as *also* $x = X$. Therefore in TQM, if detector active at laboratory time T registers a hit by a particle we *must* take the particle's position in time as $t = T$. This is required by our principle of maximum symmetry between time and space.

By the same token, in SQM if an emitter located at position X emits a particle, we take the start position of the path as $x = X$. Therefore in TQM, if an emitter active at laboratory time T emits a particle, we must take the start position in coordinate time as $t = T$.

In a practical treatment we would replace the phrases "at X" or "at T" with "within the range $X \pm \frac{\Delta X}{2}$" and "within the range $T \pm \frac{\Delta T}{2}$".

If we know both source and detector positions in space and time then all corresponding paths are clamped at both ends. In between source and detector the paths can examine all sorts of interesting times and spaces but each path is clamped at the endpoints. Alice and her dog leave from the same starting point in space time and arrive at the same ending point in space time, but while classical Alice takes the shortest path between the start and end points, *the quantum dog explores all paths.*

And this implies paths in TQM are much more complex than those in SQM. If a detector is a camera shutter, open for a fraction of a second, then any paths that arrive early or late will merely be "eaten" by the closed shutter. But what if our apparatus can somehow be toggled from transparent to absorptive and back, as via "electromagnetically induced transparency"Fleischhauer et al. [2005]? Then the paths can arrive early but then circle back, or arrive late but circle forward, or even perform a drunkard's walk around the detector till they choose to fall into it.

Since we are primarily interested in comparisons of TQM to SQM, rather than in fully exploring the elaborations of TQM, we will focus on the camera shutter model. Paths that arrive before the shutter is open or after the shutter is closed again will be silently absorbed by the camera itself. We defer to a later investigation examination of more complex paths.

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4.3.1 Metrics

With that dealt with, to compute the dispersion in time, we log how many hits we get in each time interval ("clicks per tick"):

$$\rho\left(\tau\right) \tag{156}$$

then calculate the average:

$$\langle\tau\rangle \equiv \int\limits_{-\infty}^{\infty} d\tau \rho\left(\tau\right) \tag{157}$$

and the uncertainty:

$$\langle\Delta\tau\rangle^2 \equiv \int\limits_{-\infty}^{\infty} d\tau \tau^2 \rho\left(\tau\right) - \langle\tau\rangle^2 \tag{158}$$

4.3.2 Time of arrival in SQM
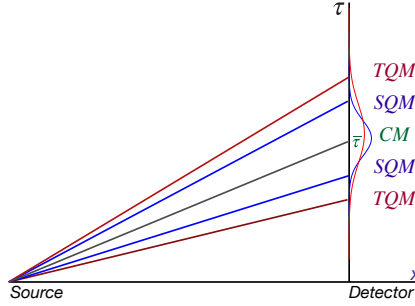
Figure 4: Time of arrival

We start with a particle with initial position $x = 0$ and with average momentum (in the x direction) of p_0. We will assume that the initial dispersion in momentum is small. We have the wave function from above. The probability density is then:

$$\bar{\rho}_\tau\left(x\right) \equiv \left|\bar{\varphi}_\tau\left(x\right)\right|^2 = \sqrt{\frac{1}{\pi\sigma_x^2\left|f_\tau^{(x)}\right|^2}} e^{-\frac{\left(x - \frac{p_0}{m}\tau\right)^2}{\sigma_x^2\left|f_\tau^{(x)}\right|^2}} \tag{159}$$

To get the dispersion in time, we integrate over possible detection times.

We expand around the time the wave function is most likely to cross the plane of the detector:

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$$\bar{\tau}_D \equiv \frac{mL_D}{p_0} = \frac{L_D}{v} \tag{160}$$

At $\bar{\tau}$ we can write x as $x = \bar{x} + \delta x$. We can turn this around, and write:

$$\tau_D = \bar{\tau}_D + \delta\tau_D \tag{161}$$

Or to lowest order in $\delta\tau$:

$$\delta x = -vd\delta\tau \tag{162}$$

We expand the numerator in the exponential in powers of $\delta\tau$:

$$\bar{\rho}_\tau(L) \approx \sqrt{\frac{1}{\pi\sigma_x^2 \left|f_\tau^{(x)}\right|^2}} e^{-\frac{(v\delta\tau)^2}{\sigma_x^2\left|f_\tau^{(x)}\right|^2}} \tag{163}$$

And the denominator to second order in powers of $\delta\tau$:

$$\sigma_x^2\left|f_\tau^{(x)}\right|^2 = \sigma_x^2 + \frac{\tau^2}{m^2\sigma_x^2} = \sigma_x^2 + \frac{(\bar{\tau}+\delta\tau)^2}{m^2\sigma_x^2} \approx \frac{(\bar{\tau}+\delta\tau)^2}{m^2\sigma_x^2} \approx \frac{\bar{\tau}^2}{m^2\sigma_x^2} \tag{164}$$

Giving:

$$\bar{\rho}_{\delta\tau} \approx \sqrt{\frac{v^2 m^2\sigma_x^2}{\pi\bar{\tau}^2}} e^{-\frac{v^2 m^2\sigma_x^2}{\bar{\tau}^2}(\delta\tau)^2} \tag{165}$$

We define an effective dispersion in time:

$$\bar{\sigma}_\tau \equiv \frac{1}{mv\sigma_x}\bar{\tau} \tag{166}$$

And the probability of detection as:

$$\bar{\rho}_{\delta\tau} = \sqrt{\frac{1}{\pi\bar{\sigma}_\tau^2}} e^{-\frac{(\delta\tau)^2}{\bar{\sigma}_\tau^2}} \tag{167}$$

This is correctly normalized to one, centered on $\tau = \bar{\tau}$, and with uncertainty:

$$\Delta\tau = \frac{1}{\sqrt{2}}\bar{\sigma}_\tau \tag{168}$$

Particularly important is the inverse dependence on the velocity. Intuitively if we have a slow moving (non-relativistic) particle, it will take a long time to pass through the plane of the detector, causing the associated uncertainty in time to be relatively large.

Comparison to a time-of-arrival operator   As a cross-check, we compare our treatment to the time-of-arrival operator analysis in Muga and Leavens Muga and Leavens [2000] (who are following Kijowski Kijowski [1974]). They give a probability density in time of:

$$\rho\left(\tau\right)=\left|\int\limits_{0}^{\infty}dp\sqrt{\frac{p}{m}}e^{-\imath\frac{p^2\tau}{2m}}\hat{\bar{\varphi}}\left(p\right)\exp\left(\imath pL\right)\right|^2+\left|\int\limits_{-\infty}^{0}dp\sqrt{\frac{-p}{m}}e^{-\imath\frac{p^2\tau}{2m}}\hat{\bar{\varphi}}\left(p\right)\exp\left(-\imath pL\right)\right|^2$$

(169)

where $\hat{\bar{\varphi}}$ is an arbitrary momentum space wave function normalized to one. Assume:

$$p_0 \gg \sigma_p$$

(170)

so:

$$p \approx p_0$$

(171)

If our wave functions are closely centered on p, the term with negative momentum can be dropped or even flipped in sign without effect on the value of the integral. Further, by comparison to the exponential part, the term under the square root is roughly constant:

$$\sqrt{\frac{p}{m}} \approx \sqrt{\frac{p_0}{m}}$$

(172)

So we may replace Muga and Leaven's expression by the simpler:

$$\rho\left(\tau\right)\approx\frac{p_0}{m}\left|\int\limits_{-\infty}^{\infty}dp\,e^{-\imath\frac{p^2\tau}{2m}}\hat{\bar{\varphi}}\left(p\right)\exp\left(\imath pL\right)\right|^2$$

(173)

As the contents of the integral are the Fourier transform of the (clock) time dependent momentum space wave function, it is the clock time dependent space wave function at $x = L$:

$$\rho\left(\tau\right)\approx\frac{p_0}{m}\left|\bar{\varphi}_\tau\left(L\right)\right|^2$$

(174)

And the rest of the analysis proceeds as above.

4.3.3   Time of arrival in TQM

It is striking that there is considerable uncertainty in time even when time is treated classically. Our hypothesized uncertainty in time will be added to this pre-existing uncertainty.

Using the time wave function from above we have for the probability density in time:

$$\tilde{\rho}_\tau(t) = \sqrt{\frac{1}{\pi \sigma_t^2 \left| f_\tau^{(t)} \right|^2}} e^{-\frac{1}{\sigma_t^2 \left| f_\tau^{(t)} \right|^2} (t-\tau)^2} \tag{175}$$

We multiply by the space part from above to get the full probability density:

$$\rho_D(t, L_D) = \tilde{\rho}_D(t) \, \bar{\rho}_D(L_D) \tag{176}$$

If t were replaced by space dimension y, we would have no doubt as to how to proceed. To get the overall uncertainty in y we would integrate over clock time

$$(\Delta y)^2 = \int dy \, (y - \bar{y})^2 \int d\tau \rho_\tau(y) \, \bar{\rho}_\tau(L_D) \tag{177}$$

Therefore we write (taking $y \to t$):

$$(\Delta t)^2 = \int dt \, (t - \bar{\tau})^2 \int d\tau \tilde{\rho}_\tau(t) \, \bar{\rho}_\tau(L_D) \tag{178}$$

This is a convolution of clock time with coordinate time. To solve we first invoke the same approximations as above:

$$\sigma_t^2 \left| f_\tau^{(t)} \right|^2 = \sigma_t^2 + \frac{\tau^2}{m^2 \sigma_t^2} = \sigma_t^2 + \frac{(\bar{\tau} + \delta\tau)^2}{m^2 \sigma_t^2} \approx \frac{(\bar{\tau} + \delta\tau)^2}{m^2 \sigma_t^2} \approx \frac{\bar{\tau}^2}{m^2 \sigma_t^2}$$
$$\tilde{\sigma}_\tau \equiv \frac{\bar{\tau}}{m \sigma_t} \tag{179}$$
$$\tilde{\rho}_\tau(t) \approx \sqrt{\frac{1}{\pi \tilde{\sigma}_\tau^2}} e^{-\frac{1}{\tilde{\sigma}_\tau^2}(t-\tau)^2}$$

So we have for the full probability distribution in t:

$$\rho_{\bar{\tau}}(t) \equiv \int d\tau \sqrt{\frac{1}{\pi \tilde{\sigma}_\tau^2}} e^{-\frac{1}{\tilde{\sigma}_\tau^2}(t-\tau)^2} \sqrt{\frac{1}{\pi \tilde{\sigma}_\tau^2}} e^{-\frac{1}{\tilde{\sigma}_\tau^2}(\tau-\bar{\tau})^2}$$
$$(\Delta t)^2 = \int dt \, (t - \bar{\tau})^2 \rho_{\bar{\tau}}(t) \tag{180}$$

The convolution over τ is trivial giving:

$$\rho_{\bar{\tau}}(t) = \sqrt{\frac{1}{\pi \sigma_\tau^2}} e^{-\frac{(t-\bar{\tau})^2}{\sigma_\tau^2}} \tag{181}$$

We therefore have for the total uncertainty at the detector:

$$\sigma_\tau^2 = \tilde{\sigma}_\tau^2 + \bar{\sigma}_\tau^2 \tag{182}$$

So the dispersion in time is the sum of the dispersions from the time and the space parts. This is intuitively reasonable. Restating the definitions for the two dispersions:

$$\bar{\sigma}_\tau^2 = \frac{\bar{\tau}^2}{m^2 v^2 \sigma_x^2}$$
$$\sigma_t^2 \approx \frac{\bar{\tau}^2}{m^2 \sigma_t^2} \tag{183}$$

From the long, slow approximation, we would expect particle wave functions to have initial uncertainties in energy/time comparable to their uncertainties in momentum/space. $\sigma_t \sim \sigma_x$. But the conventional contribution has an additional $\frac{1}{v}$ in it. Since in the non-relativistic case, $v \ll 1$ the total uncertainty will be dominated by the space part.

This helps to explain why dispersion in time has not been seen by accident. It also motivates an exploration of the relativistic case, where the effects of dispersion in time should be at least comparable to the effects of dispersion in space (section 7).

5   Semi-classical approximation

5.1   Overview

In the path integral formulation of quantum mechanics, the semi-classical interpretation provides a natural bridge between the classical and the quantum pictures. Essentially it is the stationary phase approximation applied to path integrals.

In the stationary phase approximation we expand the path integral around the path where the action is varying mostly slowly. This is the path where at each point on the path we have:

$$\frac{\delta \mathcal{L}}{\delta q} = 0 \tag{184}$$

where q stands for whatever set of coordinates we are using. This is basically the trick for integrating over a Gaussian by expanding the integral around the "hump", the spot in the integral where the Gaussian is varying most slowly. In path integrals we do this for an infinite number of coordinate points, one set of coordinates for each tick of clock time, but the idea is the same (see Zee Zee [2010] for a particularly amusing development of this idea).

But this condition is also the defining condition of the classical path. So the stationary phase approximation is an expansion around the classical path.

If we look at a river valley, the classical path corresponds to the path carved out by the moving water. The slopes of the valley around the river correspond to the quantum fluctuations around the classical path.

So standard quantum mechanics is classical mechanics plus slopes in the three space dimensions. And temporal quantum mechanics is standard quantum mechanics plus the slopes in the time dimension. TQM is to SQM – with respect to time – as SQM is to CM with respect to the three space dimensions.

In the rest of this section we will:

1. extend the usual derivation of the semi-classical approximation to include coordinate time,

2. apply the semi-classical approximation to the free case (as a calibration exercise),

3. apply it to the case of a constant magnetic field,

4. and apply it to the case of a constant electric field.

The comparison of the last two cases exposes a subtle failure of duality in SQM. In classical mechanics, the electric and magnetic fields are dual. By exchanging time and a space coordinate we can interchange results for the electric and magnetic fields. But in SQM a full interchange of time and a space coordinate is not possible because quantum fluctuations in time are not permitted. We will see this explicitly when we compare the magnetic and electric cases.

5.2 Derivation of the semi-classical approximation

We now extend the familiar stationary phase approximation in SQM path integrals to TQM.

We rewrite the coordinates in the Lagrangian in 58 as $x_j = \bar{x}_j + \delta x_j$. We define $\bar{x}$ by the requirement that the first variation with respect to x be zero:

$$
\begin{aligned}
0 = -m \left(\frac{\bar{x}_j^\mu - \bar{x}_{j-1}^\mu}{\varepsilon^2} \right) + m \left(\frac{\bar{x}_{j+1}^\mu - \bar{x}_j^\mu}{\varepsilon^2} \right) \\
- \frac{q}{2\varepsilon} \left(A^\mu \left(\bar{x}_{j-1} \right) - A^\mu \left(\bar{x}_{j+1} \right) \right) - \frac{q}{2\varepsilon} \left(\bar{x}_{j+1}^\nu - \bar{x}_{j-1}^\nu \right) \frac{\partial A_\nu (\bar{x}_j)}{\partial x_\mu^j}
\end{aligned}
\tag{185}
$$

In the continuum limit this gives the classical equations of motion (B):

$$
m \ddot{\bar{x}}_j^\mu = -q \frac{dA_j^\mu}{d\tau} + q \frac{dx_j^\nu}{d\tau} \frac{\partial A_\nu (x_j)}{\partial x_\mu^j} = -q \left(\frac{\partial A^\mu (x_j)}{\partial x_\nu^j} - \frac{\partial A_\nu (x_j)}{\partial x_\mu^j} \right) \frac{dx_j^\nu}{d\tau} = q F_\nu^\mu (x_j) \dot{x}_j^\nu
\tag{186}
$$

We therefore identify $\bar{x}$ as the classical path. The full expression for the kernel is:

$$
K_\tau (x''; x') = \int \mathcal{D}x \exp \left(\imath \varepsilon \sum_{j=1}^{j=N+1} \left(\bar{L}_j + \frac{1}{2} \frac{\partial^2 \bar{L}_j}{\partial \delta x \partial \delta x} \delta x \delta x + O \left(\delta x \right)^3 \right) \right)
\tag{187}
$$

with:

$$
\bar{L}_j \equiv -\frac{m}{2} \left(\frac{\bar{x}_j - \bar{x}_{j-1}}{\varepsilon} \right)^2 - e \frac{\bar{x}_j - \bar{x}_{j-1}}{\varepsilon} \left(\frac{A \left(\bar{x}_j \right) + A \left(\bar{x}_{j-1} \right)}{2} \right) - \frac{m}{2} \varepsilon
\tag{188}
$$

We now drop the cubic and higher terms. This is the semi-classical approximation. Including only terms through the quadratic we get:

$$
K_\tau (x''; x') \approx F_\tau (x''; x') \bar{K}_\tau (x''; x')
\tag{189}
$$

with:

$$\bar{K}_\tau \left(x''; x' \right) = \exp \left(\imath \varepsilon \sum_{j=1}^{j=N+1} \bar{L}_j \right) \rightarrow \exp \left(\imath S_\tau^{(classical)} \right) \qquad (190)$$

and fluctuation factor F:

$$F_\tau \left(x''; x' \right) \equiv \int \mathcal{D}x \exp \left(\imath \varepsilon \sum_{j=1}^{j=N+1} \frac{1}{2} \frac{\partial^2 \bar{L}_j}{\partial \delta x^\mu \partial \delta x^\nu} \delta x^\mu \delta x^\nu \right) \qquad (191)$$

The fluctuation factor can be computed in terms of the action and its derivatives. The derivation is nontrivial. It is done in the one dimensional case in Schulman Schulman [1981] and in one dimension and higher dimensions in Kleinert Kleinert [2009]. Fortunately Kleinert's derivation is independent of the number of space dimensions. We up this from three to four, treating coordinate time as a fourth space dimension x_4. With derivation complete we replace x_4 with $\imath t$.

$$F_\tau \left(x''; x' \right) = \frac{1}{\sqrt{2\pi}^4} \sqrt{\det \left(-\frac{\partial^2 S_\tau \left(x''; x' \right)}{\partial x' \partial x''} \right)} \qquad (192)$$

With final result:

$$K_\tau \left(x''; x' \right) \approx \frac{1}{\sqrt{2\pi}^4} \sqrt{\det \left(-\frac{\partial^2 S_\tau \left(x''; x' \right)}{\partial x' \partial x''} \right)} \exp \left(\imath S_\tau \left(x''; x' \right) \right) \qquad (193)$$

The $\exp \left(\imath S_\tau^{(classical)} \right)$ represents the river valley and corresponds directly to the classical case. The fluctuation factor accounts for the valley slopes, for the quantum fluctuations around the classical path.

The semi-classical approximation is exact when there are no third order terms.

5.3   Free propagator

The semi-classical approximation is arguably the simplest way to derive the free propagator. Since the free Lagrangian has no terms higher than quadratic, the semi-classical approximation is exact.

Lagrangian:

$$\mathcal{L} \left(x, \dot{x} \right) = -\frac{m}{2} \dot{x}^2 - \frac{m}{2} \qquad (194)$$

Euler-Lagrange equation:

$$m\ddot{x} = 0 \qquad (195)$$

The classical path is a straight line from start to finish:

$$x_{\tau'} = \frac{\Delta x}{\tau}\tau' + x \tag{196}$$

The Lagrangian along the classical path is a constant:

$$\mathcal{L} = -\frac{m}{2}\left(\frac{\Delta x}{\tau}\right)^2 - \frac{m}{2} \tag{197}$$

Action:

$$S = -\frac{m}{2\tau}(\Delta x)^2 - \frac{m}{2}\tau \tag{198}$$

Determinant of the action:

$$\det\left(-\frac{\partial^2 S\left(x'';x'\right)}{\partial x''\partial x'}\right) = -\frac{m^4}{\tau^4} \tag{199}$$

And full kernel:

$$K_\tau\left(x'';x'\right) = -\frac{m^2}{4\pi^2\tau^2}\exp\left(-\frac{\imath m}{2\tau}\left(x''-x'\right)^2 - \imath\frac{m}{2}\tau\right) \tag{200}$$

We can plug this into the defining equation for the kernel to confirm we have our signs and factors right:

$$\left(2m\imath\frac{\partial}{\partial\tau} - \left(\partial_\mu\partial^\mu + m^2\right)\right)K_\tau\left(x'';x'\right)\theta\left(\tau\right) = \delta\left(x''-x'\right)\delta\left(\tau\right) \tag{201}$$

A table of the free kernels is assembled in F.

5.4 Constant magnetic field

One of the advantages of TQM is a more complete symmetry between the treatments of magnetic and electric cases. We treat the magnetic field first.

Consider a constant magnetic field in the z direction: $\vec{B} = (0,0,B)$ with vector potential $\vec{A} = (0,Bx,0)$. We ignore the mass term, which can be gauged away. The Lagrangian is then:

$$L = -\frac{m}{2}\dot{t}^2 + \frac{m}{2}\dot{\vec{x}}^2 + \frac{q}{2}B\dot{y}x \tag{202}$$

The kernel splits up into t and z pieces, which are just the free kernels, and an x,y piece, which is the standard magnetic kernel:

$$K_\tau\left(t',x',y',z';t,x,y,z\right) = \tilde{K}_\tau^{(free)}\left(t';t\right)\bar{K}_\tau^{(free)}\left(z';z\right)\bar{K}_\tau^{(mag)}\left(x',y';x,y\right) \tag{203}$$

We define the Larmor frequency:

$$\omega \equiv \frac{qB}{m} \tag{204}$$

Kleinert Kleinert [2009] takes advantage of the formal resemblance of the magnetic part to the harmonic oscillator to compute the classical action:

$$S_\tau^{(mag)}\left(x',y';x,y\right) = \frac{m}{2}\left(\frac{\omega}{2}\cot\left(\frac{\omega\tau}{2}\right)\left(\left(\Delta x\right)^2 + \left(\Delta y\right)^2\right) + \omega\left(xy' - x'y\right)\right)$$
(205)

Giving fluctuation factor (in two dimensions):

$$\det\left(-\frac{\partial'\partial S_\tau^{(mag)}}{\partial x'\partial x}\right) = -\frac{\imath}{4\pi}\frac{m\omega}{\sin\left(\frac{\omega\tau}{2}\right)}$$
(206)

And full kernel:

$$K_\tau^{(mag)}\left(x',y';x,y\right) = -\frac{\imath}{4\pi}\frac{m\omega}{\sin\left(\frac{\omega\tau}{2}\right)}\exp\left(\imath S_\tau^{(mag)}\left(x',y';x,y\right)\right)$$
(207)

As a quick check, we note we recover the free form in the limit $\omega \to 0$.

Note that the TQM and SQM kernels are fully equivalent. The changes in TQM relate to the time part, which is factored out.

5.5   Constant electric field

Next consider a constant electric field in the x direction: $\vec{E} = (E, 0, 0)$ with electric potential $\Phi = -Ex$ or in four dimensional notation $A = (-Ex, 0, 0, 0)$. The Lagrangian is now:

$$L = -\frac{m\dot{t}^2}{2} + \frac{m}{2}\dot{\vec{x}}^2 + \frac{qE}{2}\dot{t}x$$
(208)

The kernel factors as before:

$$K_\tau\left(t',x',y',z';t,x,y,z\right) = \bar{K}_\tau^{(free)}\left(y',z';y,z\right)K_\tau^{(elec)}\left(t',x';t,x\right)$$
(209)

We define, analogously to the Larmor frequency:

$$\alpha \equiv \frac{qE}{m}$$
(210)

The integrals are formally equivalent to the magnetic case with the substitutions:

$$\begin{aligned}y &\to -\imath t\\ t &\to \imath y\\ \omega &\to \imath\alpha\end{aligned}$$
(211)

This will be recognized as a variation on the duality of the electric and magnetic fields, which in turn comes from the ability to rotate space and time into each other in special relativity.

Therefore we can write the action, fluctuation factor, and kernel by inspection:

$$S_\tau^{(elec)}\left(t',x';t',x'\right) = \frac{m}{2}\left(\frac{\alpha}{2}\coth\left(\frac{\alpha\tau}{2}\right)\left(-\left(\Delta t\right)^2+\left(\Delta x\right)^2\right)+\alpha\left(xt'-x't\right)\right)$$
(212)

Again the fluctuation factor for the electric part comes from the determinant of a two by two matrix:

$$\det\left(-\frac{\partial''\partial' S_{\tau''\tau'}^{(elec)}}{\partial x''\partial x'}\right) = \frac{m^2\alpha^2}{4}\left(-\coth^2\left(\frac{\alpha\Delta\tau}{2}\right)+1\right) = -\frac{m^2\alpha^2}{4\sinh^2\left(\frac{\alpha\Delta\tau}{2}\right)}$$
(213)

$$K_{\tau''\tau'}^{(elec)}\left(t'',x'';t',x'\right) = \frac{m\alpha}{4\pi\sinh\left(\frac{\alpha\Delta\tau}{2}\right)}\exp\left(\imath S_{\tau''\tau'}^{(elec)}\left(t'',x'';t',x'\right)\right)$$
(214)

The limit as $\alpha \to 0$ should give the free kernel. Again, easily verified.

There are various ways to set this problem up in SQM. But in all of them the paths must integrated only over the x coordinate and therefore may only include quantum fluctuations in one dimension rather than two.

Since the fluctuation factor for SQM for the magnetic field includes two dimensions and the fluctuation factor for SQM for the electric field includes only one dimension the two can't match. In SQM duality necessarily fails at the quantum level.

5.6 Discussion

The failure of duality in SQM is not a surprise. The duality of the electric and magnetic fields follows from special relativity. TQM by construction respects this fully; SQM does not. Given that the electric fields are the time-space components of the $F_{\mu\nu}$ tensor, while the magnetic fields correspond to the space-space components of F, it is clear that given the special handling of time in SQM means transformations between the two will be problematic.

This in turn means that duality provides another way to produce experimental tests of TQM. For instance, we could look at variations on the Aharonov-Bohm effect, especially those involving time-dependent electric fields: Aharonov and Bohm [1959], Weder [2011a,b].

6 Single and double slit experiments

6.1 Overview

The single slit in time experiment provides the decisive test of temporal quantum mechanics. In SQM, the narrower the slit, the *less* the dispersion in subsequent

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time-of-arrival measurements. In TQM, the narrower the slit, the *greater* the subsequent dispersion in subsequent time-of-arrival measurements. *In principle, the difference may be made arbitrarily great.*

This distinction follows directly from the fundamental principles of quantum mechanics. Picture a quantum wave function going through a gate in space. If the gate is wide, diffraction by the edges is minimal and the subsequent broadening of the wave function minimal. The gate will clip the beam around the edges and that will be about it. But if the gate is narrow, then the wave function will spread in a nearly circular pattern and the subsequent broadening will be arbitrarily great.

In terms of the uncertainty principle, the gate represents a measurement of the position. The narrower the gate, the less the uncertainty at the gate. If Δy is small, then Δp_y must be correspondingly large and the resulting spread greater at the detector. As $\Delta y \to 0 \Rightarrow \Delta p_y \to \infty$. But a large Δp_y implies – with a bit of time – a large spread at the detector.

We translate this from space to time. One way to visualize a gate in time is as a very fast camera shutter. The faster the shutter the smaller Δt and correspondingly the greater ΔE. The greater ΔE, the greater the dispersion in velocities and the greater the dispersion in time-of-arrival at the detector.

But – as always in this investigation – there are complicating factors.

First, in space we can make the gate entirely perpendicular to the beam. Beam traveling in x, gate in y. The beam can start with zero momentum in the y direction, letting the x momentum act as a carrier. But there is no such thing as a particle that does not have at least some momentum in time, i.e. energy. We can't achieve a complete separation between the direction of the measurement and of the beam (but see below 7.8.4).

Second, if ΔE gets large enough, we will begin to move into the relativistic zone. Our dispersion in ΔE will be curbed by relativistic constraints.

Third, to the best of our knowledge the problem of passage through a single or double slit has not even been exactly solved in SQM. The problem does not get simpler in TQM.

Finally, it is not enough to work out what the rules are for TQM; we have to do so in a way that makes comparisons to SQM straightforward.

There is already a significant literature on the "in time" versions of the single and double slit experiments. The investigation of this problem started over sixty years ago with Moshinsky Moshinsky [1951, 1952] and continues, with a recent review by Gerhard and Paulus Gerhard G. Paulus [2009]. Particularly interesting for our purposes are treatments of scattering of wave functions, as Umul Umul [2009] and Marchewka and Schuss Marchewka and Schuss [1998, 1999].

The approach we take here will be the simplest that still foregrounds the essential problems. For the single slit problem, we will focus on a beam moving from left to right in the x direction, going through a very fast camera shutter, and arriving at a detector. The distribution of detections in clock time will give us the dispersion in time-of-arrival, which is the key measurement.

We will ignore paths that loop back and forth through the gate. Most

elementary treatments make this assumption of a single passage (for an analysis of the effects of multiple passages of a gate see Yabuki, Raedt, and SawantYabuki [1986], De Raedt et al. [2012], Sawant et al. [2014]).

We will also, following Feynman and Hibbs Feynman and Hibbs [1965], take the gate as having a Gaussian shape, rather then turning on and off instantly. This results in the wave functions at the detector also being Gaussian in shape. With a sharp edged gate, the wave functions at the detector become sums of error functions, and significantly more complex.

We will treat the SQM case first, describing the beam using the p_x basis. When we turn to the TQM case we will use the SQM beam as a carrier, with the TQM wave function in time orthogonal to it. This makes the SQM/TQM comparison as direct as possible.

We conclude the analysis with an brief examination of the double slit in time. Here the differences between SQM and TQM are subtler and therefore less interesting.

6.2 Single slit with time a parameter
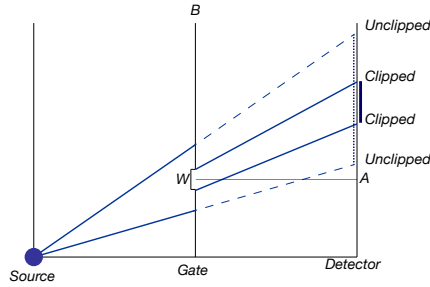
Figure 5: Single slit with time a parameter

We start by calculating the effects of a single slit in time on the dispersion of time-of-arrival measurements in SQM.

We start with a wave function in x:

$$\bar{\varphi}_\tau \left(x \right) = \sqrt[4]{\frac{1}{\pi \sigma_x^2}} \sqrt{\frac{1}{f_\tau^{(x)}}} e^{\imath p_0 x - \frac{1}{2\sigma_x^2 f_\tau^{(x)}} \left(x - x_0 - \frac{p_0}{m}\tau \right)^2 - \imath \frac{p_0^2}{2m}\tau} \tag{215}$$

or in p:

$$\hat{\bar{\varphi}}_\tau \left(p \right) = \sqrt[4]{\frac{1}{\pi \sigma_p^2}} e^{-\imath p x_0 - \frac{(p-p_0)^2}{2\sigma_p^2} - \imath \frac{p^2}{2m}\tau} \tag{216}$$

We take the velocity $v \equiv \frac{p_0}{m}$. We assume the particle is non-relativistic so that $v \ll 1$.

The gate is located at $x = B$, centered on clock time A, with width in time W:

$$\overline{G}_\tau = e^{-\frac{(\tau - A)^2}{2W^2}} \tag{217}$$

For simplicity we center the particle beam on the gate. If $\bar{\tau}_G$ is the average time at which the particle reaches the gate, we arrange it so that:

$$\bar{\tau}_G = A \tag{218}$$

This gives:

$$v = \frac{B}{A} \tag{219}$$

With the detector at position $x = L$, we define T as the average time at which the particle is detected:

$$T \equiv vL \tag{220}$$

The evolution of the wave function is simplest in the p basis, but the gate defined in τ basis. To switch back and forth between these two points of view we define:

$$\begin{aligned} p &= p_0 + \delta p = mv + \delta p \\ \tau &= \bar{\tau} + \delta\tau \end{aligned} \tag{221}$$

We assume that the incoming wave function can be treated as a sum of p rays. This worked in the time-of-arrival case, has the merit of simplicity, and lets us make a direct comparison between TQM and SQM. And this lets us change the basis from $p \to \tau$ and back using:

$$p = m\frac{x}{\tau} \tag{222}$$

We assume the beam is reasonably well-focused and use a para-axial approximation, to quadratic order:

$$\begin{aligned} \delta p &= m\frac{x}{\bar{\tau}+\delta\tau} - m\frac{x}{\bar{\tau}} \approx m\frac{x}{\bar{\tau}}\left(-\frac{\delta\tau}{\bar{\tau}} + \frac{\delta\tau^2}{\bar{\tau}^2}\right) \\ \delta\tau &= m\frac{x}{p_0+\delta p} - m\frac{x}{p_0} \approx \frac{x}{v}\left(-\frac{\delta p}{p_0} + \frac{\delta p^2}{p_0^2}\right) \end{aligned} \tag{223}$$

At the gate, the wave function in p space is:

$$\hat{\bar{\varphi}}_{G(-)}(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}}e^{-\frac{(\delta p)^2}{2\sigma_p^2} - \imath\frac{p^2}{2m}\tau_G} \tag{224}$$

In τ space:

$$\bar{\varphi}_{G(-)}(\delta\tau) = \sqrt[4]{\frac{1}{\pi\sigma_G^2}}e^{-\frac{\delta\tau^2}{2\sigma_G^2} - \imath\frac{1}{2m}\left(\frac{mB}{A+\delta\tau}\right)^2(A+\delta\tau)} = \sqrt[4]{\frac{1}{\pi\sigma_G^2}}e^{-\frac{\delta\tau^2}{2\sigma_G^2} - \imath\frac{mB^2}{2A}\left(1-\frac{\delta\tau}{A} + \frac{\delta\tau^2}{A^2}\right)} \tag{225}$$

with σ_G defined by:

$$\frac{\delta\tau}{\sigma_G}k = -\frac{\delta p}{\sigma_p} \to \sigma_G = -\frac{\delta\tau}{\delta p}\sigma_p = A\frac{\sigma_p}{p_0} \tag{226}$$

For the SQM case, we assume no diffraction in time: the wave function will be clipped by the gate, not diffracted. This means we must multiply the wave function by the gate function:

$$\bar{\varphi}_{G(+)}(\delta\tau) = e^{-\frac{(\delta\tau)^2}{2W^2}}\bar{\varphi}_{G(-)}(\delta\tau) \tag{227}$$

Now we convert back to p space, but on the far side of the gate:

$$\hat{\bar{\varphi}}_{G(+)}(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}}e^{-\frac{\delta p^2}{2\sigma_p^2}-\frac{A^2}{2W^2}\frac{\delta p^2}{p_0^2}-\imath\frac{p^2}{2m}\tau_G} \tag{228}$$

Evolve to detector:

$$\hat{\bar{\varphi}}_D(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}}e^{-\frac{\delta p^2}{2\sigma_p^2}-\frac{A^2}{2\Sigma^2}\frac{\delta p^2}{p_0^2}-\imath\frac{p^2}{2m}\tau_D} \tag{229}$$

Switch back to τ space in the same way as at the gate. The significant change is from $\sigma_p \to \sigma'_p$ where the primed dispersion in p is a kind of average of the original dispersion in p and the width of the gate:

$$\frac{1}{(\sigma'_p)^2} = \frac{1}{\sigma_p^2} + \frac{A^2}{W^2}\frac{1}{p_0^2} \tag{230}$$

With this we can write out the wave function at the detector by inspection, first in p space:

$$\hat{\bar{\varphi}}_D(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}}e^{-\frac{\delta p^2}{(\sigma'_p)^2}-\imath\frac{p^2}{2m}\tau_D} \tag{231}$$

then in τ space:

$$\bar{\varphi}_D(\delta\tau) = \sqrt[4]{\frac{1}{\pi\sigma_G^2}}e^{-\frac{\delta\tau^2}{(\sigma'_G)^2}-\imath\frac{mL^2}{2T}\left(1-\frac{\delta\tau}{T}+\frac{\delta\tau^2}{T^2}\right)} \tag{232}$$

with the primed dispersion in clock time given by:

$$\sigma'_G \equiv T\frac{\sigma'_p}{p_0} \tag{233}$$

The dispersion in clock time at the detector is basically a scaled version of the dispersion in momentum post gate.

Note the factor $\sqrt[4]{\frac{1}{\pi\sigma_G^2}}$ is unchanged. The ratio:

$$\left(\frac{\sigma'_G}{\sigma_G}\right)^2 \tag{234}$$

tells us what percentage of the particles get through the gate. But this does not effect the calculation of the dispersion, which is normalized. The width of the gate in momentum space is:

$$W_p \equiv W \frac{p_0}{A} \tag{235}$$

Therefore the effective dispersion of the wave function in clock time is:

$$(\sigma'_G)^2 = \frac{T^2}{p_0^2} \left(\frac{\sigma_p^2 W_p^2}{\sigma_p^2 + W_p^2} \right) \tag{236}$$

When the gate is wide open, the dispersion at the detector is defined by the initial dispersion of the beam:

$$W_p \to \infty \Rightarrow \sigma'_G \to \sigma_p \tag{237}$$

But when the gate is much narrower than the beam, the dispersion at the detector is defined by the width of the gate:

$$W_p \to 0 \Rightarrow \sigma'_G \to W_p \tag{238}$$

The key point is that the narrower the gate, the narrower the beam. While the approximations we have used have been simple, this is a fundamental implication of the SQM view of time, of time as classical. In SQM, wave functions are not, by assumption, diffracted by a gate, they are clipped. And therefore their dispersion in time must be reduced by the gate rather than increased by it.

6.3   Single slit with time an observable

We now have a baseline from SQM; we turn to the single slit in time in TQM using the SQM treatment as a starting point.
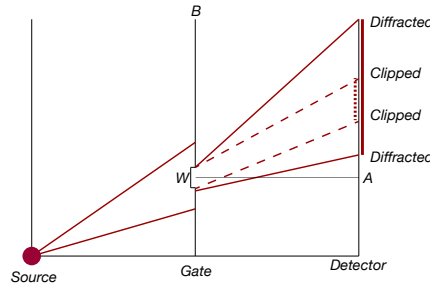

Figure 6: Single slit with time an observable

For TQM, we start with a wave function factored in time and momentum:

$$\psi_\tau (t) = \tilde{\varphi}_\tau (t) \, \hat{\tilde{\varphi}}_\tau (p) \tag{239}$$

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We are treating the p part as a carrier, using the wave function from above. The momentum part will be unchanged throughout.

By assumption, the gate will act only on the time part:

$$\widetilde{G}_t = e^{-\frac{(t-A)^2}{2W^2}} \tag{240}$$

We take the particle as starting with $t_0 = 0$. The wave function pre-gate is:

$$\tilde{\varphi}_\tau(t) = \sqrt[4]{\frac{1}{\pi\sigma_t^2}}\sqrt{\frac{1}{f_\tau^{(t)}}} e^{-\imath E_0 t + \imath \frac{E_0^2}{2m}\tau - \frac{1}{2\sigma_t^2 f_\tau^{(t)}}\left(t - t_0 - \frac{E_0}{m}\tau\right)^2} \tag{241}$$

We will again take the particle as non-relativistic: E_0 such that $\frac{E_0}{m} \approx 1$.
On arrival at the gate the wave function is:

$$\tilde{\varphi}_{G^{(-)}}(t_G) = \sqrt[4]{\frac{1}{\pi\sigma_t^2}}\sqrt{\frac{1}{f_G^{(t)}}} \exp\left(-\imath E_0 t_G - \frac{(t_G - A)^2}{2\sigma_t^2\left(1 - \imath\frac{A}{m\sigma_t^2}\right)} + \imath\frac{p_0^2}{2m}A\right) \tag{242}$$

Post gate:

$$\tilde{\varphi}_{G^{(+)}}(t_G) = e^{-\frac{(t_G-A)^2}{2W^2}}\tilde{\varphi}_{G^{(-)}}(t_G) \tag{243}$$

The effect of the gate on the wave function is to rescale it:

$$\frac{1}{2\sigma_t^2\left(1 - \imath\frac{A}{m\sigma_t^2}\right)} \rightarrow \frac{1}{2W^2} + \frac{1}{2\sigma_t^2\left(1 - \imath\frac{A}{m\sigma_t^2}\right)} \tag{244}$$

We define rescaling functions g^2 and h by:

$$\frac{1}{W^2} + \frac{1}{\sigma_t^2\left(1 - \imath\frac{A}{m\sigma_t^2}\right)} = \frac{1}{g^2\sigma_t^2\left(1 - \imath\frac{h}{m\sigma_t^2}\right)} \tag{245}$$

We have by inspection:

$$\begin{aligned} W \rightarrow 0 &\Rightarrow g \rightarrow \frac{W}{\sigma_t}, h \rightarrow 0 \\ W \rightarrow \infty &\Rightarrow g \rightarrow 1, h \rightarrow A \end{aligned} \tag{246}$$

In between these limits, we equate real and imaginary parts to get:

$$g^2(W, \sigma, A) = \frac{W^2}{\sigma_t^2} \frac{\left(\frac{A^2}{m^2} + \sigma_t^2(\sigma_t^2 + W^2)\right)}{\left(\frac{A^2}{m^2} + (\sigma_t^2 + W^2)^2\right)} \tag{247}$$

$$h(W, \sigma, A) = A \frac{\sigma_t^2 W^2}{\frac{A^2}{m^2} + \sigma_t^4 + \sigma_t^2 W^2} \tag{248}$$

This in turn gives us the t wave function at the detector. We replace σ_t and τ_G in the initial wave function by g and h:

$$\tilde{\varphi}_D\left(t_D\right) = N \sqrt[4]{\frac{1}{\pi\sigma_t^2}} \sqrt{\frac{1}{f_G}} e^{-\imath E_0 t_D - \frac{1}{2g^2\sigma_t^2 f_D'}(t_D - \bar{t}_D)^2 + \imath \frac{p_0^2}{2m}\tau_D} \tag{249}$$

$$f_D' \equiv 1 - \imath \frac{h + (T - A)}{m\sigma_t^2} \tag{250}$$

As the gate opens up entirely, as $W \to \infty$, we recover the wave function with no gate, as expected:

$$\begin{aligned} g &\to 1 \\ h + T - A &\to T \\ \bar{\varphi}_D' &\to \bar{\varphi}_D \end{aligned} \tag{251}$$

But as $W \to 0$ we have:

$$\begin{aligned} \sigma_t g &\to W \\ h &\to 0 \end{aligned} \tag{252}$$

Effectively the post gate wave function looks like an initial wave function with width in time A; the gate resets the wave function. We can reapply the analysis from above (subsection 4.3.2) to get:

$$(\Delta t)^2 = \left(\frac{1}{2W^2} + \frac{1}{2\bar{\sigma}_\tau^2}\right)\frac{\bar{\tau}^2}{m^2} \tag{253}$$

Therefore:

1. In SQM the uncertainty in clock time is proportional to the width of the gate.

2. In TQM the uncertainty in clock time is *inversely* proportional to the width of the gate.

These effects are results of the fundamental assumptions. In SQM, time is a parameter and the gate must *clip* the incoming wave function. In TQM, time is an operator and the gate must *diffract* the incoming wave function.

Therefore the observable effect may be made, in principle, *arbitrarily* large.

6.4 Double slit in time

The double slit experiment has been described by Feynman Feynman et al. [1965] as the "central mystery" of quantum mechanics. There have been many tests of the double slit experiment; the double slit in time variation has been done by Lindner et al Lindner et al. [2005].

The double slit in time experiment is less useful here: both TQM and SQM give the same spacing of peaks and valleys at the detector.

We start with the analysis in Gerhard and Paulus Gerhard G. Paulus [2009]. They consider a setup where two Gaussian wave packets are "born" at the same
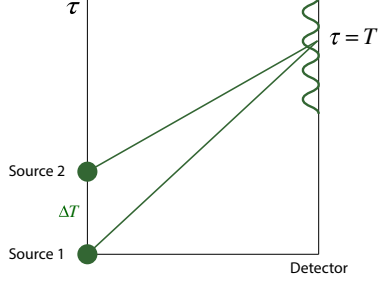
Figure 7: Double slit in time

position, but one at a time ΔT after the other. They consider wave packets with zero group velocity; we consider packets with group velocity $\frac{p_0}{m}$.

We assume the source emits a Gaussian wave packet at time zero then at time ΔT later emits a second Gaussian wave packet with the same shape but with overall phase Φ relative to the first.

Focusing on the time dependence, the two wave packets may be written in momentum space as:

$$\hat{\tilde{\varphi}}_0 \left(p \right) e^{-\imath \frac{p^2}{2m} \tau} \tag{254}$$

and:

$$\hat{\tilde{\varphi}}_{\Delta T} \left(p \right) e^{-\imath \frac{p^2}{2m} (\tau - \Delta T) - \imath \Phi} \tag{255}$$

The initial relative phase between the two is:

$$e^{\imath \frac{p_0^2}{2m} \Delta T - \imath \Phi} \tag{256}$$

The dependence of the phase on clock time for each is the same:

$$e^{-\imath \frac{p^2}{2m} \tau} \tag{257}$$

Therefore there is no additional interference in TQM beyond that resulting from the initial phase difference, already accounted for in SQM.

The interference pattern will therefore have the same spacing. Each of the peaks will be a bit wider, but that effect is already accounted for in the analysis of the time-of-arrival for a free particle. It is the single not the double slit experiment which provides the decisive test of TQM.

7 Multiple particles

"In what follows we assume that even though $A = \delta m$ is formally divergent, it is still `small´ in the sense that it is of the order of $1/137$ times the electron mass." – Sakurai Sakurai [1967]

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7.1 Why look at the multiple particle case?

In principle, we have enough to falsify TQM. Why then look at the multiple particle case?

1. From the above we see the effects of TQM will be largest at relativistic velocities and short times. This implies we need to look at high energies and, since high energies imply particle creation, at the multiple particle case. (We thank Dr. Steve Libby for bringing this point to our attention.)

2. There is the further concern that TQM may not be renormalizable. If the loop integrals in SQM are barely renormalizable, with the addition of one more dimension the loop integrals in TQM may well become completely intractable.

3. Extending TQM to the multiple particle case opens up some new effects.

4. And then there is of course the intrinsic interest of the question.

At the same time, there are formidable difficulties: the literature on quantum field theory is vast and complex (among the references we have found helpful are Bjorken and Drell [1965a,b], Roman [1969], Rivers [1987], Ramond [1990], Kaku [1993], Peskin and Schroeder [1995], Weinberg [1995a], Huang [1998], Weinberg [1995b], Itzykson and Zuber [2005], Maggiore [2005], Zee [2010], Nazarov and Danon [2013], Das [2014], Lancaster and Blundell [2014], Horwitz [2015]).

As a practical matter, we can't extend this literature from SQM to TQM in a single paper. Frankly, we doubt we could properly analyze a single realistic experiment except in a paper approaching the length of this one. Further if we tried, we might very well lose sight of the essential principles in the course of looking at detector efficiencies, recovery times, and the like.

Instead we will focus on analyzing a toy model in a series of toy situations – but in a way that will make clear how to extend the toy model to more realistic and useful cases.

To extend TQM from the single to the multiple particle case we will deploy our third requirement: consistency between the single particle and multiple particle domains.

Consider the set of Feynman diagrams associated with any specific problem. We will use the same set of Feynman diagrams in TQM as in SQM. For each diagram we will replace the SQM free propagators with TQM free propagators, replace the SQM initial wave functions with TQM wave functions, but otherwise leave matters unchanged. To compare TQM to SQM we will walk the calculations from start time to detection, looking at various questions as they arise.

With this approach the parallelism between the single and multiple particle cases is obvious: we use the same free propagator for each, but in the multiple particle case we treat a richer and more detailed set of interactions.

Since the complications of spin are inessential here, we will look at a simple model with three massive spinless particles A, B, and C (correspondingly very loosely to the electron, photon, and proton).

As in the single particle case, we will first work out the rules for the various pieces of the path integral, then apply these rules to a few simple cases. There is of course no possibility of covering all cases, but we will do enough to make clear how to extend the approach to an arbitrary problem. Steps:

1. Extend Fock space from 3D to 4D.

2. Reuse the existing field theory Lagrangian densities by interpreting the time as the coordinate time.

3. Compute the action by integrating over both clock and coordinate times, so that the integrals over the Lagrangian density goes from $\int d\tau d\vec{x} \rightarrow \int d\tau dt d\vec{x}$.

4. Verify that the free particle propagator computed with this approach matches the single particle free particle propagator.

5. Compare the SQM and TQM approaches for:

    (a) Emission of a particle,

    (b) Absorption of a particle,

    (c) Exchange of a particle,

    (d) Loop correction to the mass.

This last calculation provides a preliminary success for TQM. If treated naively, the loop diagrams in TQM are divergent. But the combination of Morlet wavelet decomposition and entanglement in time – neither alone sufficient – causes the loop integrals to converge.

As we develop the multiple particle case, we will see a number of additional effects:

1. anti-symmetry in time,

2. correlations in time (Bell's theorem in time),

3. forces of anticipation and regret.

With the approach taken here we have no free or adjustable parameters. We have a simple transition from single to multiple particle treatment. We do not have the familiar ultraviolet divergences. And we have a large number of opportunities for experimental test.

7.2 ABC model

The simplest model we can find that lets us cover the basic interactions has three spinless, massive particle species A, B, and C. We assign them non-zero masses m, μ, and M respectively. We will take μ as small as it needs to be. They are real fields. A's emit and absorb B's with amplitude λ. C's emit and
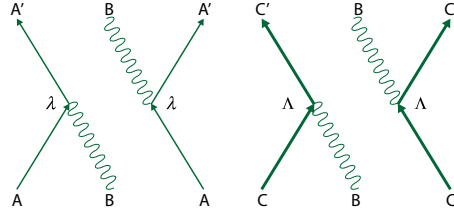
Figure 8: Interactions in the ABC model

absorb B's with amplitude Λ. A's and C's do not talk with each other directly. There are no other interactions.

Loosely, A is a spinless model of an electron, B of a photon, and C of a proton.

We will focus primarily on the A and B particles. We will need the C for the discussion of particle exchange.

7.3 Fock space

In the single particle case we generalized the wave functions from three dimensions to four. Here we generalize Fock space from three dimensions to four. As paths may be seen as a series of wave functions, one wave function per clock tick, this implicitly generalizes the associated paths from three dimensions to four as well.

We start using box normalization. The box runs from $-L \to L$ in all four coordinates: coordinate time and the three space dimensions. It is taken to extend well past the wavelets we are working with.

Using box normalization means that the Fourier transforms will be discrete. Discrete Fourier transforms are convenient for discussing various points of principle and to help in visualizing the field theory calculations. For the actual calculations we will use continuous wave functions.

In general the extrusion from three to four dimensions is straight forward; a few specific points require attention.

7.3.1 Fock space in three dimensions
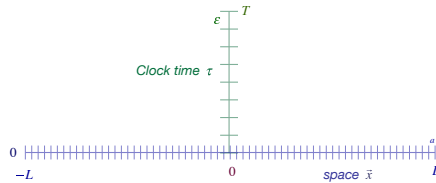
Figure 9: Fock space in three dimensions

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Single particle basis wave functions We will focus on the x coordinate here; y and z are the same.

We break our box into 2M pieces, implying a lattice spacing $a \equiv \frac{L}{M}$. a has dimensions of length.

This lets us replace the smoothly varying x with the values of x at a series of points:

$$x \in [-L, L] \rightarrow x = (-aM, \ldots, 0 \ldots, aM) \tag{258}$$

The continuous/discrete translation table is:

$$(x, y, z) \leftrightarrow (ai, aj, ak) \tag{259}$$

We are using i with a dot for the index, $\imath$ without a dot for the square root of -1.

Integrals over space go to sums over i, j, k:

$$\int_{-L}^{L} dx dy dz \leftrightarrow a^3 \sum_{i,j,k=-M}^{M} \tag{260}$$

The coordinate basis is trivial, just Kronecker δ functions:

$$\phi_{\vec{x}'}(\vec{x}) \equiv \delta^3(\vec{x} - \vec{x}') \leftrightarrow \delta_{ii'}\delta_{jj'}\delta_{kk'} \tag{261}$$

We assume the wave functions are periodic in $2L$. The periodic condition is not important; L will be chosen large enough that all interesting wave functions are well inside of it. They will be trivially periodic because they are zero on both sides of each dimension.

We normalize the basis wave functions to one:

$$\int_{-L,-L,-L}^{L,L,L} d\vec{x} \phi_{\vec{k}}^*(\vec{x}) \phi_{\vec{k}'}(\vec{x}) = \delta_{\vec{k}\vec{k}'} \tag{262}$$

Giving:

$$\phi_{\vec{k}}(\vec{x}) = \frac{1}{\sqrt{2L}^3} \exp\left(\imath \vec{k} \cdot \vec{x}\right) \tag{263}$$

Now we can expand an arbitrary wave function in terms of the basis functions:

$$\phi(\vec{x}) = \sum_{\vec{k}} c_{\vec{k}} \phi_{\vec{k}}(\vec{x}) \tag{264}$$

The measure in the path integrals is in terms of the c's, not the basis functions:

$$\mathcal{D}\phi \equiv \prod_{n=0}^{N} \mathcal{D}_n \phi, \mathcal{D}_n \phi \equiv \prod_{\vec{k}} dc_{\vec{k}} \tag{265}$$

so there is one set of space integrals at each clock tick.

At the end of the discrete part of the calculation we will be letting M, N, and L go to infinity. As the effects of TQM are averaged out over larger times we will *not* be letting T go to infinity, even in the SQM case.

In the continuum limit we have:

$$\phi_{\vec{k}}(\vec{x}) \rightarrow \frac{1}{\sqrt{2\pi}^3} \exp\left(\imath \vec{k} \cdot \vec{x}\right) \tag{266}$$

Multi-particle wave functions Things become interesting when we go to multiple particle wave functions. For two particles:

$$\phi_{\vec{k}\vec{k}'}(1,2) \equiv \frac{1}{\sqrt{2}}\left(\phi_{\vec{k}}(1)\phi_{\vec{k}'}(2) + \phi_{\vec{k}}(2)\phi_{\vec{k}'}(1)\right) \tag{267}$$

As we get to larger and larger numbers of particles, these wave functions become tricky to write out and manage. To simplify, we use the familiar annihilation and creation operators, defined by:

$$\begin{aligned} a_{\vec{k}}^\dagger \left|n_{\vec{k}}\right\rangle &= \sqrt{n_{\vec{k}} + 1}\left|n_{\vec{k}} + 1\right\rangle \\ a_{\vec{k}} \left|n_{\vec{k}}\right\rangle &= \sqrt{n_{\vec{k}}}\left|n_{\vec{k}} - 1\right\rangle \end{aligned} \tag{268}$$

with the usual commutation operators:

$$\left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] = \delta_{\vec{k}\vec{k}'} \tag{269}$$

We define the single particle operator:

$$\bar{\phi}(\vec{x}) = \sum_{\vec{k}} a_{\vec{k}} \bar{\phi}_{\vec{k}}^\dagger(\vec{x}) + a_{\vec{k}}^\dagger \bar{\phi}_{\vec{k}}(\vec{x}) \tag{270}$$

An arbitrary multiple particle basis state may be built up by products of these:

$$\left|\{n_{\vec{k}}\}\right\rangle = \frac{1}{\sqrt{\prod_{\vec{k}} n_{\vec{k}}!}} \prod_{\vec{k}} \left(a_{\vec{k}}^\dagger\right)^{n_{\vec{k}}} |0\rangle \tag{271}$$

where $\{n_{\vec{k}}\}$ is a specific set of occupation numbers.

We define the general wave function in the occupation number basis:

$$\sum_{\{n_{\vec{k}}\}} c_{\{n_{\vec{k}}\}} \left|\{n_{\vec{k}}\}\right\rangle \tag{272}$$

with normalization:

$$1 = \sum_{\{n_{\vec{k}}\}} c^2_{\{n_{\vec{k}}\}} \tag{273}$$

Note we are *not* defining the creation and annihilation operators in terms of an infinite set of harmonic oscillators; we are defining them by their effects on Fock space. We can think of movement in Fock space as a giant game of snakes and ladders, with the creation operators as the ladders, the annihilation operators as the snakes.

7.3.2 Fock space in four dimensions
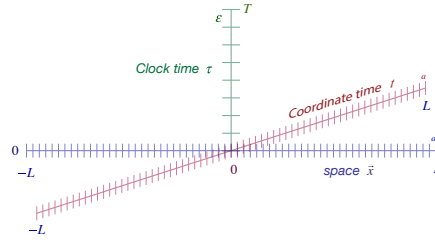
Figure 10: Fock space in four dimensions

Single particle basis wave functions The development in TQM works along the same lines but with one more dimension t and a fourth index h. We are treating coordinate time essentially like a fourth spatial dimension, much like the $x_4 \equiv \imath ct$ trick of earlier works on special relativity (or our derivation of the semi-classical approximation in 5.2). Paths in TQM will typically start at $\tau = 0$ and finish at some defined clock time $\tau = T$. But their paths in coordinate time may well dive before $t = 0$ and roam past $t = T$. To make sure that all of the relevant paths are included in our box, we require that $-L \ll 0$ and $L \gg T$. Using the same L for time as for space, and requiring that $L \to \infty$ accomplishes this.

The continuous/discrete translation table is now:

$$(t, x, y, z) \leftrightarrow (ah, ai, aj, ak) \tag{274}$$

So we have:

$$t = ah \tag{275}$$

Note there is no requirement that the lattice spacing a in coordinate time t match the step spacing ϵ in clock time τ; in fact in general we will have: $a \neq \varepsilon$.

We promote integrals and sums over three dimensions to integrals and sums over four:

$$\int d\vec{x} \to \int dt d\vec{x}, \sum_{ijk} f_{ijk} \to \sum_{hijk} f_{hijk} \tag{276}$$

$$\phi_{x'}(x) \equiv \delta^4(x - x') \leftrightarrow \delta_{hh'}\delta_{ii'}\delta_{jj'}\delta_{kk'} \tag{277}$$

The k's are again periodic. We need to think for a moment about how our wave functions get their start in life. At clock time zero, the TQM wave function will look something like:

$$\tilde{\varphi}_0(t) \sim \exp\left(-\frac{(t - t_0)^2}{2\sigma_t^2}\right) \tag{278}$$

One subtlety: we always develop our wave functions from $\tau = 0 \to \tau = T$. Consider the wave function at $\tau = 0$. How did it get there? what of its past? Won't at some time the wave function have existed before $-L$, before the box appeared? The working answer is that the shape of the wave function at $\tau = 0$ tells us all we need to know of its history before that time. If the wave function at $\tau = 0$ is well within the box, if $\sigma_t \ll L$, we have what we need. What happened in the past, stays in the past.

We normalize the basis wave functions to one:

$$\int_{-L,-L,-L,-L}^{L,L,L,L} dt d\vec{x} \phi_k^*(t, \vec{x}) \phi_{k'}(t, \vec{x}) = \delta_{kk'} \tag{279}$$

Giving:

$$\phi_k(x) = \frac{1}{4L^2} \exp(-\imath kx) \tag{280}$$

Now we can expand an arbitrary wave function in terms of the basis functions:

$$\phi(x) = \sum_k c_k \phi_k(x) \tag{281}$$

Again, the measure in the path integrals is in terms of the c's, not the basis functions:

$$\mathcal{D}\phi \equiv \prod_{n=0}^{N} \mathcal{D}_n \phi, \mathcal{D}_n \phi \equiv \prod_k dc_k \tag{282}$$

so there is one set of time and space integrals at each clock tick.

Again, at the end of the discrete part of the calculation we will be letting M, N, and L go to infinity. And we will *not* be letting T go to infinity.

In the continuum limit we have:

$$\phi_k(x) \to \frac{1}{4\pi^2} \exp\left(-\imath wt + \imath \vec{k} \cdot \vec{x}\right) \tag{283}$$

Multiple particle wave functions   Symmetrization in four dimensions works exactly as in three. For two particles we have:

$$\phi_{kk'}(1,2) \equiv \frac{1}{\sqrt{2}}\left(\phi_k(1)\phi_{k'}(2) + \phi_k(2)\phi_{k'}(1)\right) \tag{284}$$

And the creation and annihilation operators work in the same way:

$$\begin{aligned} a_k^\dagger |n_k\rangle &= \sqrt{n_k+1}\,|n_k+1\rangle \\ a_k |n_k\rangle &= \sqrt{n_k}\,|n_k-1\rangle \end{aligned} \tag{285}$$

With the commutators:

$$\left[a_k, a_{k'}^\dagger\right] = \delta_{kk'} \tag{286}$$

Single particle operator:

$$\phi(x) = \sum_k a_k \phi_k^\dagger(x) + a_k^\dagger \phi_k(x) \tag{287}$$

Arbitrary multiple particle:

$$|\{n_k\}\rangle = \frac{1}{\sqrt{\prod_k n_k!}} \prod_k \left(a_k^\dagger\right)^{n_k} |0\rangle \tag{288}$$

where $\{n_k\}$ is a specific set of occupation numbers, now over all possibilities in four dimensions.

And we again define the general wave function as a sum over all possible $\{n_k\}$:

$$\sum_{\{n_k\}} c_{\{n_k\}} |\{n_k\}\rangle \tag{289}$$

with normalization condition:

$$1 = \sum_{\{n_k\}} c_{\{n_k\}}^2 \tag{290}$$

This defines the Fock space in four dimensions, along the same lines as the one in three. We are again playing snakes and ladders, but with four dimensional snakes and ladders rather than three.

7.3.3   Anti-symmetry in time

We assume the same overall symmetry properties are required in four dimensions as in three. This implies that wave functions can use the coordinate time to help meet their symmetry responsibilities, with potentially amusing implications. In particular if the wave function is anti-symmetric in time, it will have the "wrong" symmetry properties in space.

This is testable, at least in principle.

Say we have wide wave functions in time and space $A(t)$ and $B(x)$, and narrow wave functions in time and space $a(t)$ and $b(x)$. The particles are identified as 1 and 2.

An acceptable initial wave function is:

$$\varphi_{sym}(1,2) = \frac{1}{\sqrt{2}}(A_1 B_1 a_2 b_2 + A_2 B_2 a_1 b_1) \tag{291}$$

This clearly has the right symmetry between particles 1 and 2.

We wish to break this down into sums over products of wave functions in time and space.

The symmetrical basis functions in time and space are:

$$\begin{aligned} \tilde{\varphi}_{sym}(1,2) &= \frac{1}{\sqrt{2}}(A_1 a_2 + A_2 a_1) \\ \bar{\varphi}_{sym}(1,2) &= \frac{1}{\sqrt{2}}(B_1 b_2 + B_2 b_1) \end{aligned} \tag{292}$$

If we use these as a product we get:

$$\tilde{\varphi}_{sym}(1,2)\,\bar{\varphi}_{sym}(1,2) = \frac{1}{2}(A_1 B_1 a_2 b_2 + A_1 B_2 a_2 b_1 + A_2 B_1 a_1 b_2 + A_2 B_2 a_1 b_1) \tag{293}$$

where the two middle terms do not belong.

The anti-symmetric basis functions in time and space are:

$$\begin{aligned} \tilde{\varphi}_{anti}(1,2) &= \frac{1}{\sqrt{2}}(A_1 a_2 - A_2 a_1) \\ \bar{\varphi}_{anti}(1,2) &= \frac{1}{\sqrt{2}}(B_1 b_2 - B_2 B_1) \end{aligned} \tag{294}$$

and their product is:

$$\tilde{\varphi}_{anti}(1,2)\,\bar{\varphi}_{anti}(1,2) = \frac{1}{2}(A_1 B_1 a_2 b_2 - A_1 B_2 a_2 b_1 - A_2 B_1 a_1 b_2 + A_2 B_2 a_1 b_1) \tag{295}$$

Therefore the sum of the completely symmetric and the completely anti-symmetric gives the target wave function:

$$\varphi_{sym}(1,2) = \frac{1}{2}\left(\tilde{\varphi}_{sym}(1,2)\,\bar{\varphi}_{sym}(1,2) + \tilde{\varphi}_{anti}(1,2)\,\bar{\varphi}_{anti}(1,2)\right) \tag{296}$$

To get a wave function which is completely symmetric in time and space together we need to use both the symmetric and the anti-symmetric basis functions.

7.4 Lagrangian

What should we use as a Lagrangian? We will look at this first from a classical perspective.

In SQM the Lagrangian for a massive spin 0 free particle is given by:

$$\bar{\mathcal{L}}^{free}\left[\phi,\dot{\phi}\right] = \frac{1}{2}\frac{\partial\phi}{\partial\tau}\frac{\partial\phi}{\partial\tau} - \frac{1}{2}\nabla\phi\nabla\phi - \frac{m^2}{2}\phi^2 \tag{297}$$

In classical mechanics, the wave functions may be written as sums over the basis plane waves:

$$\phi_\tau\left(\vec{x}\right) \sim \sum_{\vec{k}} c_{\tau,\vec{k}}\phi_{\tau,\vec{k}}\left(\vec{x}\right) \tag{298}$$

The action is the integral of this over space and clock time:

$$\int_0^T d\tau d\vec{x}\bar{\mathcal{L}}^{free}\left[\phi,\dot{\phi}\right] \tag{299}$$

Typically we let the limits in clock time go to $\pm\infty$, usually somewhere near the end of the analysis. Here that would average out the effects of any dispersion in time. So just as in the definition of Fock space, we keep the total clock time finite.

How to extend this Lagrangian to include coordinate time?

By our first requirement, x and t have to rotate into each other under a Lorentz transformation. The only way to do this is to change clock time to coordinate time: $\tau \to t$. So we start with:

$$\mathcal{L}^{free}[\phi] = \frac{1}{2}\partial_t\phi\partial_t\phi - \frac{1}{2}\nabla\phi\nabla\phi - \frac{m^2}{2}\phi^2 \tag{300}$$

The wave functions may be written as sums over the basis plane waves, with all the clock time dependence in the coefficients c.

$$\phi_\tau\left(t,\vec{x}\right) \sim \sum_k c_{\tau,k}\phi_k\left(t,\vec{x}\right) \tag{301}$$

Neither basis functions nor operators are functions of clock time. Therefore the Lagrangian is not.

To include the dependence on clock time we will need to include the integral over clock time from 0 to T:

$$S_0 \sim \int_0^T d\tau \int dt d\vec{x}\mathcal{L}^{free}[\varphi] \tag{302}$$

We can write the Lagrangian in momentum space as:

$$\mathcal{L}^{free} \sim \frac{w^2 - \vec{k}^2 - m^2}{2} \tag{303}$$

The integral over clock time gives:

$$\imath \int d\tau \rightarrow \imath \frac{w^2 - \vec{k}^2 - m^2}{2} \tag{304}$$

This has two problems: it is not dimensionless and it does not match the results for the single particle propagator. We can fix both by adding a factor of $\frac{1}{2m}$:

$$\hat{\bar{\mathcal{L}}}^{(free)} \rightarrow \frac{1}{2m}\left(\frac{1}{2}\partial_t\phi\partial_t\phi - \frac{1}{2}\nabla\phi\nabla\phi - \frac{m^2}{2}\phi^2\right) \tag{305}$$

This appears to give us an additional factor of $\frac{1}{2}$; in fact that will nicely cancel against a factor two created by normal ordering the operators, below.

With that noted, we will get the single particle propagator:

$$\hat{K} \sim \exp\left(\imath\frac{w^2 - \vec{k}^2 - m^2}{2m}\tau\right) \tag{306}$$

Alternatively (and a bit less obtrusively) we can absorb a factor of $\frac{1}{2m}$ into the normalization of the wave functions:

$$\int d^4x \phi_k^*(x)\phi_{k'}(x) = \frac{1}{2m}\delta_{kk'} \tag{307}$$

This parallels nicely the conventional normalization of the SQM wave functions:

$$\int d\vec{x} \bar{\phi}_{\vec{k}}^*(x)\bar{\phi}_{\vec{k}'}(x) = \frac{1}{2\omega_{\vec{k}}}\delta_{\vec{k}\vec{k}'} \approx \frac{1}{2m}\delta_{\vec{k}\vec{k}'} \tag{308}$$

given that, at least in the non-relativistic case, we have $\omega_{\vec{k}} \equiv \sqrt{m^2 + \vec{k}^2} \approx m$. For the AB particles together we have:

$$\mathcal{L}_{AB}[A, B] = \mathcal{L}^{free}[A] + \mathcal{L}^{free}[B] - \frac{\lambda}{2}ABA \tag{309}$$

And the extension to include C particles is obvious:

$$\mathcal{L}_{ABC}[A, B, C] = \mathcal{L}_{\mathcal{AB}}[A, B] + \mathcal{L}^{free}[C] - \frac{\Lambda}{2}CBC \tag{310}$$

Since there is no longer any explicit dependence on clock time in the TQM Lagrangian, the corresponding Hamiltonian is merely $-\mathcal{L}$, with the slightly disconcerting result that there are no non-trivial canonical momenta.

In SQM the next step is to promote the classical fields to operators:

$$\bar{\phi}(\vec{x}) \rightarrow \sum_{\vec{k}} a_{\vec{k}}\bar{\phi}_{\vec{k}}^\dagger(\vec{x}) + a_{\vec{k}}^\dagger\bar{\phi}_{\vec{k}}(\vec{x}) \tag{311}$$

So in TQM we do the same:

$$\phi(x) \to \sum_k a_k \phi_k^\dagger(x) + a_k^\dagger \phi_k(x) \tag{312}$$

7.5 Path integrals

With the Lagrangian defined, we can write the full kernel as:

$$K_T \equiv \int \mathcal{D}\phi \exp\left(\imath \int_0^T d\tau \int d^4x \mathcal{L}[\phi]\right) \tag{313}$$

This notation conceals much complexity. We start with the zero dimensional free case.

7.5.1 Zero dimensional free case
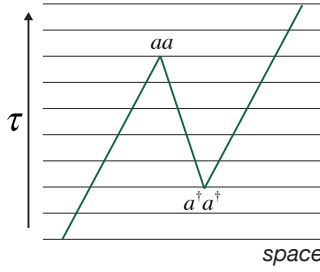
Figure 11: Zigzag paths in clock time

The basis of wave functions for the zero dimensional case is the set of possible occupation numbers from 0 to infinity. Any wave function may be written as a sum over these:

$$\psi = \sum_{l=0}^\infty c_l |l\rangle \tag{314}$$

with the normalization condition that:

$$1 = \sum_{l=0}^\infty c_l^2 \tag{315}$$

The corresponding wave function for a singleton is either just the number 1 or else $\frac{1}{\sqrt{2m}}$. If the former, then the wave function for all occupation numbers is 1. If the latter, then we have:

$$\phi_2 = \frac{1}{\sqrt{2!}}\left(\phi_1(1)\phi_1(2) + \phi_1(2)\phi_1(1)\right) = \frac{\sqrt{2!}}{2m} \tag{316}$$

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$$\phi_3 = \frac{1}{\sqrt{3!}} \left(\phi_1 \left(1 \right) \phi_1 \left(2 \right) \phi_1 \left(3 \right) + \ldots \right) = \frac{\sqrt{3!}}{\sqrt{2m}^3} \tag{317}$$

$$\phi_l = \frac{\sqrt{l!}}{\sqrt{2m}^l} \tag{318}$$

The various number states are orthogonal:

$$\langle l \mid l' \rangle = \delta_{ll'} \tag{319}$$

or:

$$\langle l \mid l' \rangle = \frac{l!}{\left(2m \right)^l} \tag{320}$$

The normalization of the wave functions will ultimately be absorbed into the definition of the c_l's, so is not that important.

The amplitude to go from one wave function to another is given by a Fock space sandwich:

$$\langle \psi' | K_T | \psi \rangle = \left\langle \sum_{l'=0}^{\infty} c_{l'} \phi_{l'} \left| \int \mathcal{D}\phi \exp \left(\imath \int_0^T d\tau \mathcal{L} \right) \right| \sum_{l=0}^{\infty} c_l \phi_l \right\rangle \tag{321}$$

With measure:

$$\mathcal{D}\phi = \prod_{n=0}^{n=N-1} \mathcal{D}\phi^{(n)}, \mathcal{D}\phi^{(n)} = \prod_{l=0}^{\infty} dc_l^{(n)} \tag{322}$$

And Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \left(a + a^\dagger \right) \phi_1 m \left(a + a^\dagger \right) \phi_1 \tag{323}$$

or if we are using the $\frac{1}{\sqrt{2m}}$ normalization:

$$\mathcal{L} = -\frac{1}{4} \left(a + a^\dagger \right) \phi_1 m^2 \left(a + a^\dagger \right) \phi_1 \tag{324}$$

In either case we finish with four terms in the Lagrangian:

$$\mathcal{L} = -\frac{m}{4} \left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger \right) \tag{325}$$

each a pair of operators. Two terms change the particle numbers; two do not. The term with two annihilation operators will reduce the number of particles by two, the term with two creation operators will increase the number of particles by two.

This means that even the simple free case has us moving up and down in Fock space. We have to deal with populations of particles.

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We could accept this and solve using coherent states, possibly with a constraint to preserve normalization. This might be more consistent with the "spirit of quantum mechanics" for which the notion of standalone particles is suspect.

But for a first investigation, we wish to develop TQM in a way that makes the closest possible connection with existing treatments, even if it is overall less satisfactory.

We proceed discretely:

$$\exp\left(\imath \int_0^T d\tau \mathcal{L}\right) \rightarrow \exp\left(\imath\varepsilon \sum_{n=1}^N \mathcal{L}\right) \tag{326}$$

and one step at a time:

$$\exp\left(\imath\varepsilon\mathcal{L}\right) \approx 1 + \imath\varepsilon\mathcal{L} \tag{327}$$

We will start with a common tactic: we will throw out all disconnected diagrams. For instance at one clock tick an $a^\dagger a^\dagger$ term could create a virtual particle/anti-particle pair which a few clock ticks later an aa then deletes. These represent self-interactions of the vacuum and are mere background noise, in common across all diagrams.

We will also use normal ordering ("always annihilate before you create"):

$$\frac{1}{2}\left(aa^\dagger + a^\dagger a\right) \rightarrow a^\dagger a + \frac{1}{2} = \hat{n} + \frac{1}{2} \tag{328}$$

where $\hat{n}$ is the number operator. The $\frac{1}{2}$ term gives us an overall constant, which we can also ignore, since it is also present for all diagrams.

Note we have just acquired a factor of two in the numerator which will cancel the $\frac{1}{2}$ we added in the last section.

There is a third problem; sometimes a particle can interact with a virtual pair. The particle is evolving in clock time and encounters a term with two annihilation operators. By chance, a particle in the vacuum encounters the same term at the same clock tick. To an outside observer, it looks as if our particle has reversed direction in clock time and is now headed backwards. This is now a connected diagram, however, so the previous rules do not exclude it.

This is however implicitly included in our analysis of particle exchange below – the middle part of the diagram is the exchanged particle. We will drop these terms unless we explicitly need them, e.g. if we are looking at pair creation or annihilation.

With this the Lagrangian reduces to a sum over number operators:

$$\mathcal{L} \rightarrow -\imath m \frac{\hat{n}}{2} = -\imath \frac{m}{2} \sum_{l=0}^\infty l\delta_{ll'} \tag{329}$$

We return the infinitesimal Lagrangian to the exponential:

$$\exp\left(-\imath\varepsilon\sum_{n=1}^{N}\frac{m}{2}\sum_{l=0}^{\infty}l\delta_{ll'}\right) \tag{330}$$

and get:

$$K_\tau\left(l;l'\right)=\exp\left(-\imath\frac{m}{2}l\tau\right)\delta_{ll'} \tag{331}$$

which makes sense as the zero dimensional matrix element: if there are l particles present, they oscillate l times as quickly as one.

7.5.2 Four dimensional free case

We now extend the zero dimensional treatment to four dimensions. Free Lagrangian:

$$\mathcal{L}^{free}\left[\phi\right]=\frac{1}{2m}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t}-\frac{1}{2m}\nabla\phi\nabla\phi-\frac{m^2}{2m}\phi^2 \tag{332}$$

where ϕ is an operator:

$$\phi\left(x\right)=\int d^4k\hat{a}_k\phi_k^\dagger\left(x\right)+\hat{a}_k^\dagger\phi_k\left(x\right) \tag{333}$$

Again the amplitude to go from one state to another is computed by constructing a Fock space sandwich:

$$A_\varepsilon=\langle\{n_{k'}\}\,|\exp\left(\imath\varepsilon\int d^4x\mathcal{L}\left[\phi,\partial\phi\right]\right)|\{n_k\}\rangle \tag{334}$$

We will use as the single particle wave functions the exponentials $\phi_k\left(x\right)\equiv\frac{1}{4\pi^2}\exp\left(-\imath kx\right)$. In momentum space the partial derivatives ∂_x turn into powers of k. We have the overall integral over x and (from the transition to momentum space) integrals over k and k' for each of the two operators in each term. The basis functions integrated over x give δ functions in k,k'. The integral over k' gives an integral over k. We are left with something that looks like:

$$\int d^4x\phi\phi\to\int d^4k\left(a_ka_{-k}+a_ka_k^\dagger+a_k^\dagger a_k+a_k^\dagger a_{-k}^\dagger\right) \tag{335}$$

This is the same as the zero dimensional case with an index k. All four terms conserve momentum, but as before two terms change the particle numbers, two do not.

We define the associated frequency:

$$f_k\equiv-\frac{w^2-\vec{k}^2-m^2}{2m} \tag{336}$$

The momentum space integral of the Lagrangian is:

$$-\imath \int_0^\tau d\tau' f_k n_k \rightarrow -\imath f_k \tau \tag{337}$$

giving the kernel for one frequency as:

$$K_T \sim \exp\left(-\imath f_k n_k \tau\right) \tag{338}$$

If $n_k \rightarrow 1$ we have the TQM single particle propagator:

$$\hat{K}_\tau\left(k; k'\right) = \exp\left(-\imath f_k \tau\right) \delta\left(k - k'\right) \tag{339}$$

Basically we are taking the existing Lagrangian and stretching it along clock time.

We have played a bit fast and loose with normalizations, but the principles are clear.

7.5.3 Measure

Fock space consists of products of basis wave functions:

$$\left|\{n_k\}\right\rangle \rightarrow \left|n_0 k_0\right\rangle \left|n_1 k_1\right\rangle \left|n_2 k_2\right\rangle \ldots \tag{340}$$

A path is a series of positions in this Fock space. The measure weights each possible position equally, so the measure is:

$$\mathcal{D}\phi \equiv \prod_{n=0}^{N-1} \mathcal{D}_n \phi, \mathcal{D}_n \phi \equiv \prod_{k,n_k} dc_{k,n_k}^{(n)} \tag{341}$$
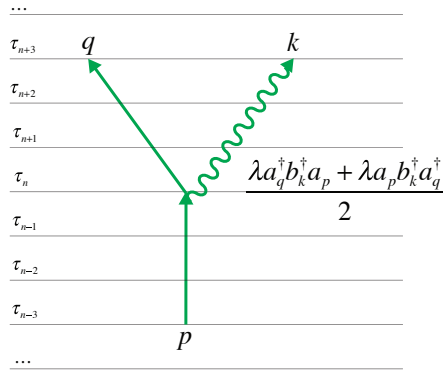
7.5.4 Interaction terms

Figure 12: Interaction term in ABC model

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Consider the discretized form of the path integral. Consider an expansion of the path integral in powers of the coupling constant. Consider the term for the first power of the coupling constant. The coupling term is represented by a term of the form:

$$\exp\left(-\imath\varepsilon\lambda\int d^4x\frac{ABA}{2}\right) \qquad (342)$$

This can happen at any of the $N-1$ intermediate steps. Assume we are looking at the case where this happens at clock tick number n. Note – in striking contrast to the SQM case – the interaction term has no dependence on n, on the clock time.

As we take the limit as $N\to\infty$ the size of the time slice $\varepsilon\equiv T/N$ goes to zero so we can approximate the exponential by:

$$1-\imath\varepsilon\lambda\int d^4x\frac{ABA}{2} \qquad (343)$$

The path integral is formed by doing the integrals from $0\to n-1$, then the integral over the interaction term at n, then the integrals from $n+1\to N-1$. The terms before n are included in the free propagator(s) from 0 to n; the terms after n are included in the free propagators from n to N, we have the interaction term to consider here.

The integral over space at step n will give us a δ function in momentum at step n: $\delta\left(k+q-p\right)$. Notice that four momentum is conserved at the vertex. This is another point of difference with SQM. In SQM only the three momentum is conserved at a vertex, the conservation of energy comes from the integral over the clock time.

For the nth time in our path integral from 0 to T, we role the dice in our game of snakes and ladders. Spelled out in terms of a and b operators we have:

$$
\begin{aligned}
A_p &= a_p\phi_p^\dagger + a_p^\dagger\phi_p \\
B_k &= b_k\phi_k^\dagger + b_k^\dagger\phi_k \\
A_q &= a_q\phi_q^\dagger + a_q^\dagger\phi_q
\end{aligned}
\qquad (344)
$$

We might drop down one step in terms of A particles with momentum p while going up one step for A particles with momentum q and one step for B particles with momentum k. This would be accomplished by a term of the form $\lambda a_q^\dagger b_k^\dagger a_p$. There are two such terms in the interaction term, neatly canceling out the factor of $\frac{1}{2}$. To contribute to the first order perturbation diagram the interaction must hit exactly once on the way from 0 to N. The result, pulling all this together, and going from discrete to continuous form is:

$$\hat{\psi}_T\left(q,k\right) = -\imath\lambda\int_0^T d\tau\int dp\hat{K}_{T\tau}^{(m)}\left(q\right)\hat{K}_{T\tau}^{(\mu)}\left(k\right)\delta\left(q+k-p\right)\hat{K}_\tau^{(m)}\left(p\right)\hat{\varphi}\left(p\right) \quad (345)$$

7.5.5 Full propagator

We now have what is required to compute an arbitrary propagator. The topology of the diagrams is unchanged from SQM: we get exactly the same set of diagrams, but with the intermediate integrals and initial wave function(s) over four dimensions rather than three. To help see this, take a standard Feynman diagram in SQM. It has an integral over clock time from start to finish (we are looking at finite clock time measurements, not beams or averages). At each vertex we have an integral over three space or three momentum coordinates, depending on our representation. Lets say we are working in the position representation. As we did in the derivation of the semi-classical approximation, add to the x_1, x_2, x_3 an x_4. Clearly the topology is unchanged. Now rotate $x_4 \rightarrow \imath t$. We have arrived at TQM, but have not changed the topology.

The path integral in perturbation expansion is given by the sum over all diagrams consistent with the boundary conditions. We have to integrate over time and space coordinates, convolute over clock time. The general propagator is given by:

$$\langle \{n_{k'}\} | \int \mathcal{D}\phi \exp\left(\imath \int_0^T d\tau \mathcal{L}^{free}[\phi] - V[\phi] \right) | \{n_k\}\rangle \tag{346}$$

where $\{n_k\}$ is the initial element in Fock space (or more generally a sum over such elements) and $\{n_{k'}\}$ is the final element in Fock space. The Feynman diagrams are generated by expanding this in powers of the coupling constant.

7.6 Free particles

How does the free propagator in TQM compare to the free propagator in SQM?
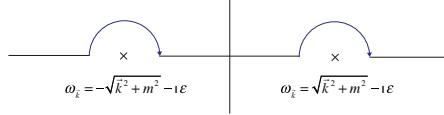
Figure 13: Contour for retarded propagator

We compute the free propagators for SQM and then TQM. We work these out for an A particle; the B and C are the same.

Our goal here to establish clearly the relationship between the SQM and TQM propagators, to make an apples-to-apples comparison between the two. The best way to do this is to look not just at the propagators but as usual at their effects on Gaussian test functions.

We will take as the starting point the respective differential equations for the SQM and TQM propagators. These may be derived from the path integral approach using the powerful generating approaches described in for instance

Kashiwa and Zee Kashiwa et al. [1997], Zee [2010]. Here we take them as a given of the analysis.

7.6.1 Free particle in SQM

We start with the Klein-Gordon equation. We define the propagator by:

$$\left(-\frac{\partial^2}{\partial\tau^2} + \nabla^2 - m^2\right)\bar{K}_\tau\left(\vec{x};\vec{x}'\right) = \imath\delta\left(\tau\right)\delta^3\left(\vec{x} - \vec{x}'\right) \tag{347}$$

In momentum space we have:

$$\frac{-\imath}{k_0^2 - \vec{k}^2 - m^2} = \frac{1}{(2\pi)^4} \tag{348}$$

We choose retarded boundary conditions. This implies that both poles have a small negative imaginary part:

$$k_0 = \pm\omega_{\vec{k}} - \imath\varepsilon, \omega_{\vec{k}} \equiv \sqrt{m^2 + \vec{k}^2} \tag{349}$$

and the inverse Fourier transform is:

$$\bar{K}_\tau\left(\vec{x},\vec{x}'\right) = -\imath\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^4}\int dk_0 d\vec{k}\frac{\exp\left(-\imath k_0\tau + \imath\vec{k}\cdot\left(\vec{x} - \vec{x}'\right)\right)}{(k_0 + \imath\varepsilon)^2 - \vec{k}^2 - m^2} \tag{350}$$

Doing the k_0 integral explicitly we get:

$$\bar{K}_\tau\left(\vec{x},\vec{x}'\right) = \frac{1}{(2\pi)^3}\int d\vec{k}\frac{\exp\left(-\imath\omega_{\vec{k}}\tau\right) - \exp\left(\imath\omega_{\vec{k}}\tau\right)}{2\omega_{\vec{k}}}\exp\left(\left(\imath\vec{k}\cdot\left(\vec{x} - \vec{x}'\right)\right)\right) \tag{351}$$

In terms of the conventional basis functions:

$$\bar{\phi}_{\vec{k}}\left(\vec{x}\right) \equiv \frac{1}{\sqrt{2\omega_{\vec{k}}}}\exp\left(\imath\vec{k}\cdot\vec{x}\right) \tag{352}$$

this is:

$$\bar{K}_\tau\left(\vec{x},\vec{x}'\right) = \frac{1}{(2\pi)^3}\int d\vec{k}\bar{\phi}_{\vec{k}}^*\left(\vec{x}\right)\bar{\phi}_{\vec{k}}\left(\vec{x}'\right)\left(\exp\left(-\imath\omega_{\vec{k}}\tau\right) - \exp\left(\imath\omega_{\vec{k}}\tau\right)\right) \tag{353}$$

To make a closer comparison to TQM we wish to shift to the basis functions:

$$\phi_{\vec{k}}\left(\vec{x}\right) \equiv \frac{1}{\sqrt{2\pi}^3}\exp\left(\imath\vec{k}\cdot\vec{x}\right) \tag{354}$$

This will let us use our familiar Gaussian test functions on the right, without worrying about how to describe them in terms of the $\frac{1}{\sqrt{2\omega}}$ basis functions used in most treatments. (Or how to undescribe them on the far side of the calculation.)

To do this we transform the conventional kernel to this basis by multiplying it by:

$$2\omega_{\vec{k}} \tag{355}$$

This gives:

$$\bar{K}_\tau\left(\vec{x},\vec{x}'\right) \to \int d\vec{k}\, \frac{\exp\left(\imath\vec{k}\cdot\vec{x}\right)}{\sqrt{2\pi}^3} \frac{\exp\left(-\imath\vec{k}\cdot\vec{x}'\right)}{\sqrt{2\pi}^3} \left(\exp\left(-\imath\omega_{\vec{k}}\tau\right) - \exp\left(\imath\omega_{\vec{k}}\tau\right)\right) \tag{356}$$

This make sense: we associate with each normalized plane wave a corresponding frequency in clock time. We have a traditional kernel of the form:

$$K_\tau = \sum_n \phi_n^* \phi_n \exp\left(-\imath\omega_n\tau\right) \tag{357}$$

We apply this to a Gaussian test function. We choose one centered on $\vec{k}_0$ with $\vec{k} \equiv \vec{k}_0 + \delta\vec{k}$, initial position $\vec{x}_0$. We will take one which is separable in the three space directions:

$$\hat{\bar{\varphi}}_0\left(\vec{k}\right) = \hat{\bar{\varphi}}_0^{(y)}\left(k_y\right) \hat{\bar{\varphi}}_0^y\left(k_y\right) \hat{\bar{\varphi}}_0^{(z)}\left(k_z\right) \tag{358}$$

where the x Gaussian test function is:

$$\hat{\bar{\varphi}}_0^{(x)}\left(k_x\right) = \sqrt[4]{\frac{1}{\pi\hat{\sigma}_x^2}}\, e^{-\imath k_x x_0 - \frac{\left(k_x - k_x^{(0)}\right)^2}{2\hat{\sigma}_x^2}} = \sqrt[4]{\frac{1}{\pi\hat{\sigma}_x^2}}\, e^{-\imath k_x x_0 - \frac{\delta k_x^2}{2\hat{\sigma}_x^2}} \tag{359}$$

and y, z the same.

The concession to relativity is to include both positive and negative frequencies for each wave vector.

We now take as a working assumption that our incoming wave function is dominated by the positive frequency part. (This is the same trick we used in the analysis of the time-of-arrival measurements in sub-section 4.3). If we were going to examine phenomena like Zitterbewegung we would need to relax this assumption.

We therefore simplify our kernel to:

$$\bar{K}_\tau\left(\vec{x},\vec{x}'\right) \to \int d\vec{k}\, \frac{\exp\left(\imath\vec{k}\cdot\vec{x}\right)}{\sqrt{2\pi}^3} \frac{\exp\left(-\imath\vec{k}\cdot\vec{x}'\right)}{\sqrt{2\pi}^3} \exp\left(-\imath\omega_{\vec{k}}\tau\right) \tag{360}$$

The momentum space form is:

$$\hat{\bar{K}}_\tau\left(\vec{k},\vec{k}'\right) = \exp\left(-\imath\omega_{\vec{k}}\tau\right) \delta^3\left(\vec{k} - \vec{k}'\right) \theta\left(\tau\right) \tag{361}$$

We next expand $\omega_{\vec{k}}$ in powers of the kinetic energy:

$$\omega_{\vec{k}} \approx m + \frac{\vec{k}^2}{2m} - \frac{\vec{k}^4}{8m^{3/2}} + O\left(\vec{k}^6\right) \tag{362}$$

If we need the relativistic corrections we can keep the fourth and higher order terms. Alternatively we could take advantage of the fact that all our calculations start with Gaussian test functions and expand, not around $\vec{k} = 0$, but around $\vec{k} = \vec{k}_0$:

$$\omega_{\vec{k}} = \sqrt{m^2 + \left(\vec{k}_0 + \delta\vec{k}\right)^2} \approx \omega_0 + \frac{\delta\vec{k} \cdot \vec{k}_0}{\omega_0} + \frac{\omega_0}{2}\left(\frac{\delta\vec{k} \cdot \delta\vec{k}}{\omega_0^2} - \frac{\left(\delta\vec{k} \cdot \vec{k}_0\right)^2}{\omega_0^4}\right) + \dots \tag{363}$$

In either case we would entangle the three momenta with each other. For now we confine ourselves to the zeroth and quadratic terms.

Applied to the Gaussian test function we get:

$$\hat{\tilde{\varphi}}_\tau\left(\vec{k}\right) = \int d\vec{k}' \hat{\tilde{K}}_\tau\left(\vec{k}; \vec{k}'\right) \hat{\tilde{\varphi}}_0\left(\vec{k}\right) \tag{364}$$

or:

$$\hat{\tilde{\varphi}}_\tau\left(\vec{k}\right) = \exp\left(-\imath m\tau - \imath\frac{\vec{k}^2}{2m}\tau\right) \hat{\tilde{\varphi}}_0\left(\vec{k}\right) \tag{365}$$

We see on the SQM side the relationship between the various levels of analysis: the particle at rest (m term), the particle moving slowly ($\frac{k^2}{2m}$) term, and the particle moving relativistically (k^4 and higher corrections).

We summarize the SQM propagator (as applied to a Gaussian test function) as:

$$\hat{\tilde{K}}_\tau\left(w, \vec{k}\right) = \exp\left(\imath m\tau + \imath\frac{\vec{k}^2}{2m}\tau + \imath\frac{O\left(\vec{k}^4\right)}{8\sqrt{m}^3}\tau\right) \tag{366}$$

7.6.2 Free particle in TQM

We have the free propagator from above:

$$\hat{K}_\tau\left(\vec{k}\right) = \exp\left(\imath\frac{m}{2}\tau - \imath\frac{w^2}{2m}\tau + \imath\frac{\vec{k}^2}{2m}\tau\right) \tag{367}$$

There are three differences between the TQM and SQM propagators. The first two are not that useful for our purposes; the third is critical.

First we have an overall factor of:

$$\exp\left(\imath\frac{m}{2}\tau\right) \tag{368}$$

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as compared to the factor of:

$$\exp\left(-\imath m\tau\right) \tag{369}$$

in the SQM propagator.

The frequency associated with, say, the mass of an electron is:

$$\frac{m_e}{\hbar} \approx \frac{.51 \cdot 10^6 eV}{6.6 \cdot 10^{-16} eV \sec} = 7.7 \cdot 10^{-24} \sec^{-1} \tag{370}$$

This is of order the frequencies associated with Zitterbewegung, about one million times the frequencies we are dealing with here. In fact, as Zitterbewegung has not itself been measured, we would have no starting point on the SQM side. We will therefore ignore this factor.

The relativistic correction factor:

$$\exp\left(\imath \frac{O\left(\vec{k}^4\right)}{8\sqrt{m}^3}\tau\right) \tag{371}$$

is present in SQM but not in TQM.

These represent higher order corrections. As TQM is predicting significant differences from SQM in even the non-relativistic case, relativistic corrections to SQM are not needed (and clutter up the analysis).

The key difference is the factor of:

$$\exp\left(-\imath \frac{w^2}{2m}\tau\right) \tag{372}$$

This represents the extension of the function in time/energy. Basically the TQM propagator is the SQM propagator with additional fuzziness in time/energy. This difference is the focus of attention in this work.

So we will take the SQM propagator as:

$$\hat{\bar{K}}_\tau\left(\vec{k}\right) = \exp\left(\imath \frac{\vec{k}^2}{2m}\tau\right) \tag{373}$$

the TQM propagator as:

$$\hat{K}_\tau\left(\vec{k}\right) = \exp\left(-\imath \frac{w^2}{2m}\tau + \imath \frac{\vec{k}^2}{2m}\tau\right) = \exp\left(-\imath \frac{w^2}{2m}\tau\right)\hat{\bar{K}}_\tau\left(\vec{k}\right) \tag{374}$$

We apply this to a Gaussian test function. We use a Gaussian test function in energy times the previous Gaussian test function in three momentum.

$$\hat{\varphi}_0\left(w, \vec{k}\right) = \hat{\bar{\varphi}}_0\left(w\right)\hat{\bar{\varphi}}_0\left(\vec{k}\right) \tag{375}$$

with energy part:

$$\hat{\tilde{\varphi}}_0 \left(w \right) \equiv \sqrt[4]{\frac{1}{\pi \hat{\sigma}_t^2}} e^{\imath w t_0 - \frac{(w - w_0)^2}{2 \hat{\sigma}_t^2}} \tag{376}$$

From the entropic analysis above (subsection 4.1):

$$\begin{aligned} w_0 &\sim \omega_{\vec{k}} \\ \hat{\sigma}_t^2 &\sim \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_t^2 \end{aligned} \tag{377}$$

Application of the TQM kernel to the Gaussian test function is trivial:

$$\hat{\varphi}_\tau \left(k \right) = \exp \left(\imath \frac{k^2 - m^2}{2m} \tau \right) \hat{\varphi}_0 \left(k \right) \tag{378}$$

7.7 Emission of a particle

What does the emission of a particle look like in TQM?
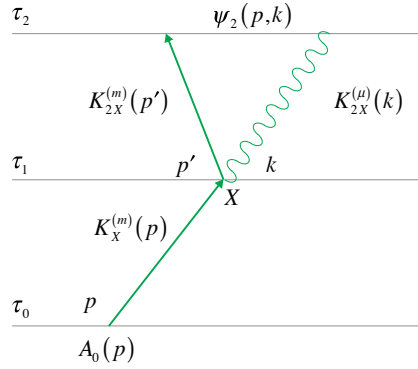
Figure 14: An A particle emits a B particle

7.7.1 Overview

We look at the case where an A particle emits a B. The initial particle expectation and dispersion are given; we wish to compute the outgoing particle expectations and dispersions.

We start with a Gaussian test function A, extended in space for the SQM case, and in time and space for TQM.

The initial expectation and dispersion are given at clock time τ_0. We wish to compute the final expectations and dispersions at clock time τ_2. The B has an amplitude λ to be emitted at each intermediate clock time τ_1.

In a first order perturbation expansion we would integrate over intermediate clock times τ_1. But here we will focus on a smaller piece of the puzzle, looking

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at the contribution to the final wave function from a single point in clock time. We look at:

$$\psi_2\left(p, k\right) = -\imath\lambda \int dp K_{21}^{(m)}\left(p'\right) K_{21}^{(\mu)}\left(k\right) \delta\left(p' + k - p\right) K_{10}^{(m)}\left(p\right) A_0\left(p\right) \tag{379}$$

We look at this first in SQM then in TQM.

7.7.2 Emission of a particle in SQM

We start our A particle as a Gaussian test function in momentum space:

$$\hat{\tilde{A}}_0\left(\vec{p}\right) = \hat{\tilde{\varphi}}_a^{(x)}\left(p_y\right) \hat{\tilde{\varphi}}_a^{(y)}\left(p_x\right) \hat{\tilde{\varphi}}_a^{(z)}\left(p_x\right) \tag{380}$$

We are marking the constants specific to the wave function A with the letter a, threading the a through the wave function. For the x component we have:

$$\hat{\tilde{\varphi}}_a^{(x)}\left(p_x\right) \equiv \sqrt[4]{\frac{1}{\pi\hat{\sigma}_x^2}} e^{-\imath p_x x_a - \frac{\left(p_x - p_x^{(a)}\right)^2}{2\hat{\sigma}_x^2}} \tag{381}$$

with y, z in parallel. The full wave function at τ_0 is:

$$\hat{\tilde{A}}_0\left(\vec{p}\right) = \sqrt[4]{\frac{1}{\pi^3 det\left(\hat{\tilde{\Sigma}}\right)}} e^{-\imath\vec{p}\cdot\vec{x}_a - \frac{1}{2}\Delta\vec{p}\cdot\hat{\tilde{\Sigma}}^{-1}\cdot\Delta\vec{p}} \tag{382}$$

with ancillary definitions:

$$\Delta\vec{p} \equiv \vec{p} - \vec{p}_a$$
$$\hat{\tilde{\Sigma}} \equiv \begin{pmatrix} \hat{\sigma}_x^2 & 0 & 0 \\ 0 & \hat{\sigma}_x^2 & 0 \\ 0 & 0 & \hat{\sigma}_x^2 \end{pmatrix} \tag{383}$$

We are dropping the parts of the kernel that depend on the rest mass. For the A particle these are independent of the interaction and therefore irrelevant. For the B particle the $exp\left(-\imath\mu\tau\right)$ is not independent of the interaction, but in the limit as $\mu \to 0$ this factor is constant and therefore also irrelevant.

The kernel that carries A from $\tau_0 \to \tau_X$ is:

$$\hat{\tilde{K}}_X^{(m)}\left(\vec{p}\right) = \exp\left(-\imath\frac{\vec{p}^2}{2m}\tau_X\right) \tag{384}$$

So the A wave function at X is:

$$\hat{\tilde{A}}_X\left(\vec{p}\right) = \sqrt[4]{\frac{1}{\pi^3 det\left(\hat{\tilde{\Sigma}}\right)}} e^{-\imath\vec{p}\cdot\vec{x}_a - \frac{1}{2}\Delta\vec{p}\cdot\hat{\tilde{\Sigma}}^{-1}\cdot\Delta\vec{p} - \imath\frac{\vec{p}^2}{2m}\tau_X} \tag{385}$$

The integral over the δ function at τ_X gives:

$$\hat{\tilde{\psi}}_2 \left(\vec{p}', \vec{k} \right) = -\imath \lambda \hat{\tilde{K}}_{2X}^{(m)} \left(\vec{p}' \right) \hat{\tilde{K}}_{2X}^{(\mu)} \left(\vec{k} \right) \hat{\tilde{A}}_X \left(\vec{p}' + \vec{k} \right) \qquad (386)$$

In a certain sense, we are pushing the final momentum back into the initial wave function. The post interaction wave function has the same shape as the initial wave function, but the shape is now shared by two particles.

The post-vertex kernels are:

$$\hat{\tilde{K}}_{2X}^{(m)} \left(\vec{p}' \right) = \exp\left(-\imath \frac{(\vec{p}')^2}{2m} \tau_{2X} \right), \hat{\tilde{K}}_{2X}^{(\mu)} \left(\vec{k} \right) = \exp\left(-\imath \frac{\vec{k}^2}{2\mu} \tau_{2X} \right) \qquad (387)$$

with $\tau_{2X} \equiv \tau_2 - \tau_X$.

The wave function at τ_2 is therefore:

$$\hat{\tilde{\psi}}_2 \left(\vec{p}', \vec{k} \right) = \sqrt[4]{\frac{1}{\pi^3 det\left(\hat{\tilde{\Sigma}} \right)}} e^{\left(-\imath \left(\vec{p}' + \vec{k} \right) \cdot \vec{x}_a - \frac{1}{2} \Delta \vec{p}' \cdot \hat{\tilde{\Sigma}}^{-1} \cdot \Delta \vec{p}' - \imath \Omega_X \tau_{2X} - \imath \Omega_0 \tau_X \right)} \qquad (388)$$

with change in momentum:

$$\Delta \vec{p}' \equiv \vec{p}' + \vec{k} - \vec{p}_a \qquad (389)$$

and initial and final energies:

$$\Omega_0 \equiv \frac{\left(\vec{p}' + \vec{k} \right)^2}{2m}, \Omega_X \equiv \frac{(\vec{p}')^2}{2m} + \frac{\vec{k}^2}{2\mu} \qquad (390)$$

We can see that an integral over τ_X would tend to subtract out components where $\Omega_0 \neq \Omega_X$, giving us an effective δ function in the SQM energy.

The final wave function is strongly correlated between left and right. The conservation condition at the vertex means A', B are each sharing part of the same initial momentum. They are like Siamese twins – separated at birth but still connected. This is the source of the mysterious spooky action at a distance complained of in the initial EPR paper Einstein [1935].

7.7.3 Emission of a particle in TQM

We take the same basic approach, but now with the coordinate energy/coordinate time included, and with the conservation condition at the vertex being for four momentum rather than three momentum.

We write the initial wave function as direct product of time and space parts: $A = \tilde{A}\bar{A}$.

$$\hat{A}_0 (p) = \hat{\tilde{A}}_a (E) \hat{\tilde{A}}_a (\vec{p}) \qquad (391)$$

The three momentum part is the same as in SQM. For the energy part we have:

$$\hat{\tilde{A}}_a \left(E\right) \equiv \sqrt[4]{\frac{1}{\pi\hat{\sigma}_t^2}}\, e^{\imath E t_a - \frac{(E - E_a)^2}{2\hat{\sigma}_t^2}} \tag{392}$$

For a first attack we need to estimate the starting position, energy, and dispersion in time. We will take:

$$\begin{aligned} t_a &\approx \tau_0 = 0 \\ E_a &\approx \sqrt{m^2 + \vec{p}_a^2} \\ \hat{\sigma}_E^2 &\approx \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2 \end{aligned} \tag{393}$$

We write the entire wave function function as:

$$\hat{A}_0 \left(p\right) = \sqrt[4]{\frac{1}{\pi^4 \det\left(\hat{\Sigma}\right)}}\, e^{-\imath p x_a - \frac{1}{2}\Delta p \hat{\Sigma}^{-1}\Delta p} \tag{394}$$

with ancillary definitions:

$$\begin{aligned} \Delta p &\equiv p - p_a \\ \hat{\Sigma} &\equiv \begin{pmatrix} \hat{\sigma}_t^2 & 0 & 0 & 0 \\ 0 & \hat{\sigma}_x^2 & 0 & 0 \\ 0 & 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & 0 & \hat{\sigma}_z^2 \end{pmatrix} \end{aligned} \tag{395}$$

The kernel that carries A from $\tau_0 \rightarrow \tau_X$ is:

$$K_X^{(m)}\left(p\right) = \exp\left(-\imath f_p \tau_X\right) = \hat{\tilde{K}}_X^{(m)}\left(E\right) \hat{\tilde{K}}_X^{(m)}\left(\vec{p}\right) \exp\left(-\imath \frac{m}{2}\tau_X\right) \tag{396}$$

with $f_p \equiv -\frac{E^2 - \vec{p}^2 + m^2}{2m}$ and with energy part:

$$\hat{\tilde{K}}_X^{(m)}\left(E\right) \equiv \exp\left(\imath \frac{E^2}{2m}\tau_X\right) \tag{397}$$

So the A wave function at X is:

$$\hat{A}_X \left(p\right) = \sqrt[4]{\frac{1}{\pi^4 \det\left(\hat{\Sigma}\right)}}\, e^{-\imath p \cdot x_a - \frac{1}{2}\Delta p \hat{\Sigma}^{-1}\Delta p}\, e^{-\imath f_p \tau_X} \tag{398}$$

The energy part at A is now:

$$\hat{\tilde{A}}_X \left(E\right) = \sqrt[4]{\frac{1}{\pi\hat{\sigma}_t^2}}\, e^{-\imath E t_a - \frac{(E - E_a)^2}{2\hat{\sigma}_t^2}}\, e^{\imath \frac{E_a^2}{2m}\tau_X} \tag{399}$$

The integral over the δ function at X gives:

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$$\hat{\psi}_2\left(p', k'\right) = -\imath\lambda\hat{K}_{2X}^{(m)}\left(p\right)\hat{K}_{2X}^{(\mu)}\left(k\right)\hat{A}_X\left(p\right)\left(p' + k\right) \tag{400}$$

Again we are pushing the sum of the final momenta back into the initial wave function. And again the post interaction wave function has the same shape as the initial wave function, but the shape is now shared by two particles. But in TQM, the shared shape is a shape in four rather than just three dimensions.

Post vertex kernels:

$$\hat{K}_{2X}^{(m)}\left(p\right) = \exp\left(-\imath f_{p'}\tau_{2X}\right), \hat{K}_{2X}^{(\mu)}\left(k\right) = \exp\left(-\imath f_k\tau_{2X}\right) \tag{401}$$

So the wave function at τ_2 is:

$$\hat{\psi}_2\left(p', k\right) = \sqrt[4]{\frac{1}{\pi^4 \det\left(\hat{\Sigma}\right)}} e^{\left(\imath(p'+k)x_a - \frac{1}{2}\Delta p'\hat{\Sigma}^{-1}\Delta p' - \imath F_X\tau_{2X} - \imath F_0\tau_X\right)} \tag{402}$$

with change in four momentum:

$$\Delta p' \equiv p' + k - p_a \tag{403}$$

and initial and final clock frequencies:

$$\begin{aligned} F_0 &\equiv -\frac{\left(E'+w\right)^2 - \left(\vec{p}+\vec{k}\right)^2 - m^2}{2m}, \\ F_X &\equiv f_{p'} + f_k = -\frac{\left(E'\right)^2 - \left(\vec{p}'\right)^2 - m^2}{2m} - \frac{w^2 - \vec{k}^2 - \mu^2}{2\mu} \end{aligned} \tag{404}$$

Per the long, slow approximation, we expect that both F_0 and F_X will be small. As noted in the free particle section most of the dependence on clock time will be carried by the coordinate time part of the wave function.

The parts dependent on the rest masses do not play a critical role, for the same reasons as in the SQM case.

The energy part is:

$$\hat{\tilde{\psi}}_2\left(E', w\right) = \sqrt[4]{\frac{1}{\pi\hat{\sigma}_t^2}} e^{\imath\left(E'+w\right)t_a - \frac{\left(E'+w-E_a\right)^2}{2\hat{\sigma}_t^2} + \imath\left(\frac{\left(E'\right)^2}{2m} + \frac{w^2}{2m}\right)\tau_{2X} - \imath\frac{\left(E'+w\right)^2}{2m}\tau_X} \tag{405}$$

The left and right halves are sharing the same shape – now also extended in energy/time – even though with increasing clock time they are separated by greater and greater distances. Again, they are like Siamese twins separated at birth but still connected, now across time as well as space.

7.7.4 Discussion

With TQM, to the correlations in three-momentum complained of in the initial EPR paper we add correlations in energy. These provide raw material for a Bell's theorem "in time".

Presumably Einstein would still be unhappy about the "spooky action at a distance", but perhaps he would be partly consoled by the inclusion of time on the same basis as space.

7.8 Absorption of a particle

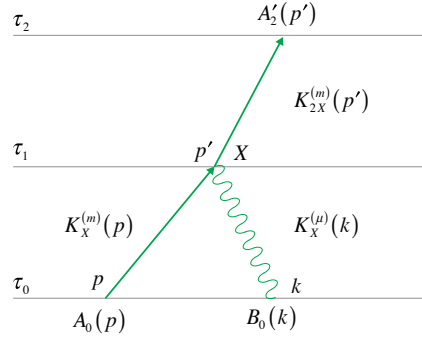What does the absorption of a particle look like in TQM?

Figure 15: An A particle absorbs a B particle

7.8.1 Overview

We look at the case where an A absorbs a B. The initial particle expectations and dispersions are given; we wish to compute the outgoing particle's expectation and dispersion.

We start with two Gaussian test functions A, B. These are centered on momenta p_a, k_b with initial expectations for position x_a, x_b. We define $p'_a \equiv p_a + k_b$.

For simplicity we take the A particle as coming in from the left and the B as coming in from the right. Without loss of generality we can assume both are coming in along the x-axis with relative offset b along the y axis. With a slight loss of generality we will assume $b \to 0$.

We have starting velocities, $v > 0, u > 0$:

$$\vec{p}_a = mv\hat{x}, \vec{k}_b = -\mu u\hat{x} \tag{406}$$

and starting points on left and right:

$$\vec{x}_a = -l\hat{x}, \vec{x}_b = d\hat{x} \tag{407}$$

In first order perturbation theory we would compute the final amplitude by integrating over all intermediate clock times τ_1. But – as with emission – we focus on the contribution to this of the interaction at a specific clock time

τ_X. Since we have a natural clock time to work with – the time defined by the intersection of the classical paths of A, B – we will use that.

This is defined by:

$$x = -l + v\tau_X = d - u\tau_X \tag{408}$$

giving crossing time τ_X:

$$\tau_X = \frac{d+l}{v+u} \tag{409}$$

and crossing position x_X:

$$x_X = \frac{vd - ul}{v+u} \tag{410}$$

Trivially $y_X = z_X = 0$.

With these simplifications we will be left with one integral to do, a convolution of the initial momenta:

$$\hat{A}_2(p') = -\imath\lambda \int dk \hat{K}_{2X}^{(m)}(p') \hat{A}_X(p' - k) \hat{\bar{B}}_X(k) \tag{411}$$

We will first treat the SQM case, then TQM.

In both cases the treatment is essentially a wrapper for a simple idea: when two particles combine to form a third the dispersion of the resulting particle will be given by a kind of average of the dispersions of the two incoming particles:

$$\frac{1}{\sigma_{a'}} \sim \frac{1}{\sigma_a} + \frac{1}{\sigma_b} \tag{412}$$

In particular if one of the incoming particles is markedly narrower than the other – let's say it is B ($\sigma_b \ll \sigma_a$) – then the dispersion of the outgoing particle will be dominated by the dispersion of the B: $\sigma'_a \approx \sigma_b$. Effectively the interaction with B will act as a measurement of the position of A at the time of interaction. The narrower B the less the uncertainty in the position of A at τ_X. A narrow B acts as a gate, decreasing the uncertainty in the position of A – and increasing the uncertainty in the corresponding momentum.

This is true for space in SQM; true for time and space for TQM. This gives us a way to extend the single slit experiment to the realm of field theory.

To be sure this simple idea requires a fair quantity of wrapper to make it precise, but that is all that is going on here.

7.8.2 Absorption of a particle in SQM

Initial wave functions The initial particles are given by:

$$\hat{\bar{A}}_0(\vec{p}) = \sqrt[4]{\frac{1}{\pi^3 det(\hat{\bar{\Sigma}})}} e^{-\imath \vec{p}\cdot\vec{x}_a - \frac{1}{2}\Delta\vec{p}\cdot\hat{\bar{\Sigma}}^{-1}\cdot\Delta\vec{p}}$$
$$\hat{\bar{B}}_0(\vec{k}) = \sqrt[4]{\frac{1}{\pi^3 \det(\hat{\bar{S}})}} e^{-\imath \vec{k}\cdot\vec{x}_b - \frac{1}{2}\Delta\vec{k}\cdot\hat{\bar{S}}^{-1}\cdot\Delta\vec{k}} \tag{413}$$

with expectations and dispersions:

$$\Delta \vec{p} \equiv \vec{p} - \vec{p}_a, \Delta \vec{k} \equiv \vec{k} - \vec{k}_b$$
$$\hat{\bar{\Sigma}} \equiv \begin{pmatrix} \hat{\sigma}_x^2 & 0 & 0 \\ 0 & \hat{\sigma}_x^2 & 0 \\ 0 & 0 & \hat{\sigma}_x^2 \end{pmatrix}, \hat{\bar{S}} \equiv \begin{pmatrix} \hat{s}_x^2 & 0 & 0 \\ 0 & \hat{s}_y^2 & 0 \\ 0 & 0 & \hat{s}_z^2 \end{pmatrix} \tag{414}$$

In momentum space the kernels from start to X are:

$$\hat{\bar{K}}_X^{(m)}(\vec{p}) = \exp\left(-\imath \frac{\vec{p}^2}{2m}\tau_X\right), \hat{\bar{K}}_X^{(\mu)} = \exp\left(-\imath \frac{\vec{k}^2}{2\mu}\tau_X\right) \tag{415}$$

The wave functions at X are therefore:

$$\hat{\bar{A}}_X(\vec{p}) = \sqrt[4]{\frac{1}{\pi^3 det\left(\hat{\bar{\Sigma}}\right)}} e^{-\imath \vec{p}\cdot\vec{x}_a - \frac{1}{2}\Delta\vec{p}\cdot\hat{\bar{\Sigma}}^{-1}\cdot\Delta\vec{p} - \imath \frac{\vec{p}^2}{2m}\tau_X}$$
$$\hat{\bar{B}}_X(\vec{k}) = \sqrt[4]{\frac{1}{\pi^3 \det\left(\hat{\bar{S}}\right)}} e^{-\imath \vec{k}\cdot\vec{x}_b - \frac{1}{2}\Delta\vec{k}\cdot\hat{\bar{S}}^{-1}\cdot\Delta\vec{k} - \imath \frac{\vec{k}^2}{2\mu}\tau_X} \tag{416}$$

Interaction The final wave function at τ_2 will be given by a convolution of all possible incoming momenta:

$$\hat{\bar{A}}_2(\vec{p}') = -\imath\lambda\hat{K}_{2X}^{(m)}(\vec{p}')\int d\vec{k}\hat{A}_X\left(\vec{p}' - \vec{k}\right)\hat{B}_X\left(\vec{k}\right) \tag{417}$$

We first focus on the convolution and specifically on the x part of the integral in momentum space:

$$\hat{I}_X^{(x)}(p'_x) \equiv \int dk_x \hat{A}_X(p'_x - k_x)\hat{B}_X(k_x) \tag{418}$$

Since this is the integral of a Gaussian it can be solved exactly. However we will get more insight by shifting to the position basis. The convolution in momentum space becomes a multiplication in position space:

$$I_X^{(x)}(x) = \sqrt{2\pi}A_X(x)B_X(x) \tag{419}$$

The Gaussian test functions for A and B are centered on their corresponding classical paths; if we didn't know about the classical paths we could have computed the intersection point by looking for the clock time where both wave functions are centered on the same value of x.

The close correspondence of classical and quantum trajectories is an attractive feature of the approach here; we can think of a particle as traveling along a classical line with quantum fuzz around it.

At the crossing time, the coordinate forms for A and B at X are therefore:

$$A_X(x) = F_X^{(x)} e^{\imath p_x^{(a)}(x-x_a) - \frac{1}{2\sigma_x^2 f_X^{(x)}}(x-x_X)^2 - \imath \frac{p_x^{(a)2}}{2m}\tau_X}$$
$$B_X(x) = G_X^{(x)} e^{\imath k_x^{(b)}(x-x_b) - \frac{1}{2s_x^2 g_X^{(x)}}(x-x_X)^2 - \imath \frac{k_x^{(b)2}}{2\mu}\tau_X} \tag{420}$$

The dispersion factors f, g and normalization factors F, G are spelled out in a A.3.1.

We define effective crossing times τ_X^* and dispersions $\hat{\sigma}_x^{*2}$ via:

$$\frac{1}{\sigma_x^2 + \imath \frac{\tau_X}{m}} + \frac{1}{s_x^2 + \imath \frac{\tau_X}{\mu}} = \frac{1}{\hat{\sigma}_x^{*2} + \imath \frac{\tau_X^{(x)*}}{m}} \tag{421}$$

A' will start at X as a particle with effective dispersion $\hat{\sigma}_x^{*2}$ originating from a time τ_X^* seconds back. We can compute the effective dispersion and effective clock time by comparing the real and imaginary parts of the defining equation.

$$\begin{aligned} \sigma_x^{*2} &= \frac{\sigma_x^2 s_\tau^{(x)4} + s_x^2 \sigma_\tau^{(x)4}}{\Sigma_\tau^{(x)4}} \\ \frac{\tau_X^{(x)*}}{m} &= \frac{\tau_X}{\Sigma_\tau^{(x)4}} \left(\frac{s_\tau^{(x)4}}{m} + \frac{\sigma_\tau^{(x)4}}{\mu} \right) \end{aligned} \tag{422}$$

$\sigma_\tau^{(x)2}$ and $s_\tau^{(x)2}$ are the absolute values of the complex dispersions:

$$\sigma_\tau^{(x)2} \equiv \sqrt{\sigma_x^4 + \frac{(\tau_X)^2}{m^2}}, s_\tau^{(x)2} \equiv \sqrt{s_x^4 + \frac{(\tau_X)^2}{\mu^2}} \tag{423}$$

and $\Sigma_\tau^{(x)2}$ is a kind of average of these:

$$\Sigma_\tau^{(x)2} \equiv \sqrt{(\sigma_x^2 + s_x^2)^2 + \tau_X^2 \left(\frac{m\mu}{m + \mu} \right)^2} \tag{424}$$

We can see from the defining equation that if either A or B has a significantly smaller absolute dispersion the effective dispersion and effective clock time will be dominated by that side. Take $s_x^2 \ll \hat{\sigma}_x^2$. If we also have that $\mu \ll m$, then we have the much simpler:

$$\begin{aligned} \hat{\sigma}_x^{*2} &\approx s_x^2 \\ \tau_X^{(x)*} &\approx 0 \end{aligned} \tag{425}$$

In this case B acts to reset A, creating an A' which is effectively starts fresh at τ_X with dispersion taken from B. B therefore acts precisely like a single slit with width s_x. This was why we chose to parameterize the interaction using the effective dispersion and clock time.

Continuing:

$$I_X^{(x)}(x) = N_X^{(x)} \varphi_X^{(*)}(x) \tag{426}$$

with a starred Gaussian test function in x:

$$\varphi_X^{(*)}(x) = F_\tau^{(x)*} e^{\imath p_x^{(a')}(x - x_X) - \frac{1}{2\sigma_x^{*2} f_X^{*(x)}}(x - x_X)^2 - \imath \frac{p_x^{(a')2}}{2m} \tau_X^{(x)*}} \tag{427}$$

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and an overall constant $N_X^{(x)}$ independent of x.[1] The overall constant will drop out when we calculate the final expectation and dispersion, so the effective dispersion and crossing time carry all the physically significant information.

Final wave function   Since all x dependence is carried by $\varphi_X^{(*)}(x)$, we have the Fourier transform by inspection:

$$\hat{I}_X^{(x)}(p'_x) = N_X^{(x)}\hat{\varphi}_X^{(*)}(p'_x) \tag{428}$$

with the momentum space form of the starred wave function:

$$\hat{\bar{\varphi}}_X^*(p'_x) = \sqrt[4]{\frac{1}{\pi\hat{\sigma}_X^{*2}}}e^{-\imath p'_x x_X - \frac{\left(p'_x - p_x^{(a')}\right)^2}{2\hat{\sigma}_X^{*2}} - \imath\frac{\vec{p}'^2}{2m}\tau_X^{(x)*}} \tag{429}$$

The other two dimensions work in parallel. Now that we are back in momentum space we have the resulting wave function at τ_2:

$$\hat{\bar{A}}_2(\vec{p}') = -\imath\lambda\exp\left(-\imath\frac{\vec{p}'^2}{2m}\tau_{2X}\right)N_X^{(x)}N_X^{(y)}N_X^{(z)}\hat{\bar{\varphi}}_X^*(\vec{p}') \tag{430}$$

For our purposes the most interesting aspect is the associated uncertainty in momentum. The overall normalization drops out:

$$\left\langle\left(\vec{p}' - \vec{p}^{(a')}\right)^2\right\rangle = \frac{\int d\vec{p}'\left(\vec{p}' - \vec{p}^{(a')}\right)^2\left|\hat{\bar{\varphi}}_X(\vec{p}')\right|^2}{\int d\vec{p}'\left|\hat{\bar{\varphi}}_X(\vec{p}')\right|^2} \tag{431}$$

The uncertainty in p is defined by the post-interaction wave function. For the x direction this is:

$$\hat{\sigma}_X^{*2} = \frac{1}{\sigma_X^{*2}} \tag{432}$$

We can read off the physically important part of the resulting wave function from the starred dispersion. If we have $\mu \ll m, s_x \ll \sigma_x$ then if we detect A' at all, we know to within $s_x \ll \sigma_x$ where A was at τ_X. But this A' has momentum p'_x with a large effective dispersion $\hat{\sigma}_x^* \sim 1/s_x$. So we lose all knowledge of A's original momentum p_x; the highly variable p'_x will hide that from us.

We see in detail how the Heisenberg uncertainty principle works in the x, p_x dimension.

---

[1] For the record: $N_X^{(x)} \equiv \sqrt{2\pi}\frac{F_\tau^{(x)}G_\tau^{(x)}}{F_\tau^{(x)*}}e^{-\imath p_x^{(a)}(x_a - x_X) - \imath k_x^{(b)}(x_b - x_X) + \imath\frac{p_x^{(a')2}}{2m}\tau_X^* - \imath\left(\frac{p_x^{(a)2}}{2m} + \frac{k_x^{(b)2}}{2\mu}\right)\tau_X}$.

7.8.3 Absorption of a particle in TQM

We work along the same lines to extend the analysis to TQM. The classical trajectories now include a time component:

$$t = t_0 + \gamma\tau \tag{433}$$

We will treat the time parts of the wave functions as non-relativistic, $\gamma \approx 1$. This is consistent with our use of the non-relativistic approximation for the space parts.

We will assume our initial wave functions are centered on τ_0:

$$t_0 = \tau_0 \Rightarrow t_0 = 0 \tag{434}$$

As a result we have the same intersection point in coordinate time that we have in clock time:

$$t_X = \tau_X \tag{435}$$

Initial wave functions For the initial wave functions we have A and B as products of their time and space parts:

$$
\begin{aligned}
\hat{A}_0\left(p\right) &= \sqrt[4]{\frac{1}{\pi det(\hat{\Sigma})}}e^{-\imath p x_a - \frac{1}{2}\Delta p \hat{\Sigma}\Delta p} = \hat{\tilde{A}}_0\left(E\right)\hat{\tilde{A}}_0\left(\vec{p}\right) \\
\hat{B}_0\left(k\right) &= \sqrt[4]{\frac{1}{\pi^4\det(\hat{S})}}e^{\imath k x_b - \frac{1}{2}\Delta k \hat{S}^{-1}\Delta k} = \hat{\tilde{B}}_0\left(w\right)\hat{\tilde{B}}_0\left(\vec{k}\right)
\end{aligned}
\tag{436}
$$

with time parts:

$$
\begin{aligned}
\hat{\tilde{A}}_0\left(E\right) &= \sqrt[4]{\frac{1}{\pi\hat{\sigma}_t^2}}e^{\imath E t_a - \frac{(E - E_a)^2}{2\hat{\sigma}_t^2}} \\
\hat{\tilde{B}}_0\left(w\right) &= \sqrt[4]{\frac{1}{\pi\hat{s}_t^2}}e^{\imath w t_b - \frac{1}{2\hat{s}_t^2}(w - w_b)^2}
\end{aligned}
\tag{437}
$$

and:

$$
\begin{aligned}
&\Delta p \equiv p - p_a, \Delta k \equiv k - k_b \\
&\hat{\Sigma} \equiv \begin{pmatrix} \hat{\sigma}_t^2 & 0 & 0 & 0 \\ 0 & \hat{\sigma}_x^2 & 0 & 0 \\ 0 & 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & 0 & \hat{\sigma}_z^2 \end{pmatrix}, \hat{S} \equiv \begin{pmatrix} \hat{s}_t^2 & 0 & 0 & 0 \\ 0 & \hat{s}_x^2 & 0 & 0 \\ 0 & 0 & \hat{s}_y^2 & 0 \\ 0 & 0 & 0 & \hat{s}_z^2 \end{pmatrix}
\end{aligned}
\tag{438}
$$

and expectations and dispersions in time/energy:

$$
\begin{aligned}
&t_a \approx \tau_0 = 0, t_b \approx \tau_0 = 0 \\
&E_a \approx \sqrt{m^2 + \vec{p}_a^2}, w_b \approx \sqrt{\mu^2 + \vec{k}_b^2} \\
&\hat{\sigma}_E^2 \approx \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2, \hat{s}_t^2 \approx \hat{s}_x^2 + \hat{s}_y^2 + \hat{s}_z^2
\end{aligned}
\tag{439}
$$

The wave functions at X are:

$$\hat{A}_X(p) = \sqrt[4]{\frac{1}{\pi^4 det(\hat{\Sigma})}} e^{\imath p \cdot x_a - \frac{1}{2}\Delta p \hat{\Sigma}^{-1}\Delta p} e^{-\imath f_p \tau_X}$$
$$\hat{B}_X(k) = \sqrt[4]{\frac{1}{\pi^4 \det(\hat{S})}} e^{\imath k x_b - \frac{1}{2}\Delta k \hat{S}^{-1}\Delta k - \imath f_k \tau_X} \tag{440}$$

Interaction To compute the wave function at τ_2 we again convolute the incoming wave functions:

$$\hat{A}_2(p') = -\imath \lambda \hat{K}_{2X}^{(m)}(p') \int dk \hat{A}_X(p' - k)\hat{B}_X(k) \tag{441}$$

The coordinate energy part of the integral in momentum space is:

$$\hat{\tilde{I}}_X(E') \equiv \int dw \hat{\tilde{A}}(E' - w)\hat{\tilde{B}}(w) \tag{442}$$

In coordinate time:

$$\tilde{I}_X(t) = \sqrt{2\pi}\tilde{A}_X(t)\tilde{B}_X(t) \tag{443}$$

The wave functions in time are:

$$\tilde{A}_X(t) = F_X^{(t)} e^{-\imath E_a t - \frac{1}{2\sigma_t^2 f_X^{(t)}}\left(t - t_a - \frac{E_a}{m}\tau_X\right)^2 + \imath \frac{E_a^2}{2m}\tau_X}$$
$$\tilde{B}_X(t) = G_X^{(t)} e^{-\imath w_b t - \frac{1}{2s_t^2 g_X^{(t)}}\left(t - t_b - \frac{w_b}{\mu}\tau_X\right)^2 + \imath \frac{w_b^2}{2\mu}\tau_X} \tag{444}$$

The quadratic arguments of the Gaussians both reduce to $(t - \tau_X)^2$ in our non-relativistic approximation.

By the same analysis in t as in x we get:

$$\tilde{I}_X(t) = \tilde{N}_X \tilde{\varphi}_X^{(*)}(t) \tag{445}$$

with starred wave function [2]:

$$\tilde{\varphi}_X^{(*)}(t) \equiv F_X^{(t)*} e^{\imath E'_a(t - t_X) - \frac{1}{2\sigma_t^{*2} f_X^{*(t)}}(t - t_X)^2 - \imath \frac{E'_a{}^2}{2m}\tau_X^{(t)*}} \tag{446}$$

with effective dispersion and clock time as before with $x \to t$:

$$\sigma_t^{*2} = \frac{\sigma_t^2 s_\tau^{(t)4} + s_t^2 \sigma_\tau^{(t)4}}{\Sigma_\tau^{(t)4}}$$
$$\frac{\tau_X^{(t)*}}{m} = \frac{\tau_X}{\Sigma_\tau^{(t)4}}\left(\frac{s_\tau^{(t)4}}{m} + \frac{\sigma_\tau^{(t)4}}{\mu}\right) \tag{447}$$

As with space, we take $s_t^2 \ll \hat{\sigma}_t^2$ and $\mu \ll m$ getting in the limit:

---

[2] And – by analogy with space – overall constant $\tilde{N}_X \equiv$ $\sqrt{2\pi}\frac{F_X^{(t)}G_X^{(t)}}{F_X^{(t)*}} e^{\imath E_a(t_a - t_X) + \imath w_b(t_b - t_X) - \imath \frac{E'_a{}^2}{2m}\tau_X^* + \imath\left(\frac{E_a{}^2}{2m} + \frac{w_b^2}{2\mu}\right)\tau_X}$.

$$\hat{\sigma}_t^{*2} \approx s_t^2$$
$$\tau_X^{(t)*} \approx 0 \tag{448}$$

So in time as in space: B acts to reset A, creating an A' which effectively starts fresh at τ_X with dispersion taken from B. B therefore acts precisely like a single slit – in time – with width s_t.

Since all t dependence is carried by $\tilde{\varphi}_X^{(*)}(t)$, we have the Fourier transform by inspection:

$$\hat{I}_X^{(x)}(p'_x) = \tilde{N}_X \hat{\tilde{\varphi}}_X^{(*)}(E') \tag{449}$$

with the momentum space form of the starred wave function:

$$\hat{\tilde{\varphi}}_X^{(*)}(E') = \sqrt[4]{\frac{1}{\pi \hat{\sigma}_X^{(t)*2}}} e^{\imath E' x_X - \frac{(E' - E'_a)^2}{2\hat{\sigma}_X^{(t)*2}} + \imath \frac{(E')^2}{2m} \tau_X} \tag{450}$$

Final wave function

$$\hat{A}_2(p') = \hat{\tilde{A}}_2(E') \hat{\tilde{A}}_2(\vec{p}')$$
$$\hat{\tilde{A}}_2(E') = \exp\left(\imath \frac{E'^2}{2m} \tau_{2X}\right) N_X^{(t)} \hat{\tilde{\varphi}}_X(E') \tag{451}$$

Uncertainty in time For our purposes the most interesting aspect is the associated uncertainty in energy. For this the overall normalization drops out:

$$\left\langle \left(E' - E^{(a')}\right)^2 \right\rangle = \frac{\int dE' \left(E - E^{(a')}\right)^2 \left|\hat{\tilde{\varphi}}_X(E)\right|^2}{\int dE' \left|\hat{\tilde{\varphi}}_X(E)\right|^2} \tag{452}$$

The uncertainty in E is defined by the post-interaction wave function:

$$\hat{\sigma}_t^{*2} = \frac{1}{\sigma_t^{*2}} \tag{453}$$

7.8.4   Discussion

There are obviously a variety of ways to improve the quality of the approximations used here. A good start might be to integrate over the intermediate τ_1, perhaps using a stationary phase approximation around τ_X. But the fundamental principles of the analysis will not be changed by this and in fact are clearer in the simpler case.

Two principal effects   We have as noted in the introduction two principal effects:

1. Increased dispersion in time at all points.

2. An uncertainty principle in time/energy on a completely equivalent basis to one for space/momentum.

Anticipation and regret   With respect to the increased dispersion, there is a further point, not explored in detail here, but interesting.

If wave functions are extended in time as they are in space, then in a collision they will start to interact earlier, cease interacting later than would otherwise be the case.

The resultant broadening of the interaction zone may be seen as representing forces of anticipation and regret.

Two different measurements   With respect to the uncertainty principle, we have shown in detail how the uncertainty principle for time/energy works exactly as the uncertainty principle for space/momentum.

However there is a problem in testing this with time-of-arrival measurements. If we use time-of-arrival measurements we are measuring dispersion in time, yes, but we are also measuring dispersion in momentum along the axis of flight. Since we expect the initial dispersions in time to be of order of those in space, it may be difficult to *prove* there was no dispersion in time. Perhaps it was lost in the error bars?

The basic problem is the time-of-arrival measurement is being used to measure two different things: one measurement cannot serve two masters.

One way to separate the two measurements would be to run the post-interaction particle through a magnetic field. Let's say the particle will be bent to the right by the magnetic field. The y position will serve as usual as a measurement of velocity. But if at each y position we also record the time-of-arrival the time-of-arrival should now serve as a measurement of dispersion in coordinate time.

If we graph the clicks on a y, τ grid, the faster particles will hit earlier in time and more to the left. In SQM we would expect to see a relatively narrow trace from small y, τ to large y, τ. In TQM we would expect to see the same trace on average, but significantly broader in τ at each y.

7.9   Exchange of a particle

What does the exchange of a particle look like in TQM?

7.9.1   Overview

We look at the case where an A and a C exchange a B. The initial particle expectations and dispersions are given; we wish to compute the outgoing particle
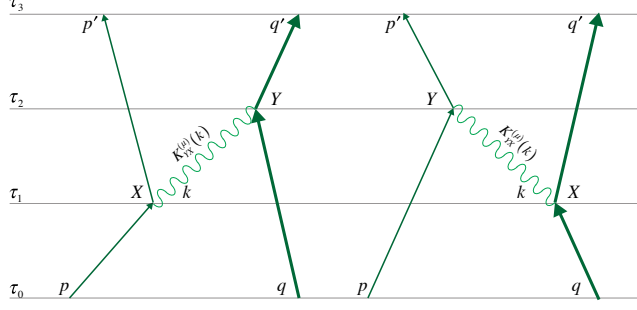
Figure 16: An A particle exchanges a B particle with a C particle

expectation and dispersions. We have two cases: the A emits a B which is then absorbed by C and the C emits a B which is then absorbed by the A. We will call these the left and right cases.

To lowest order in perturbation expansion we have convolutions over the two intermediate clock times τ_1, τ_2.

$$
\hat{\psi}_3^{(left)}(p', q') = -\lambda\Lambda \int_0^{\tau_3} d\tau_2 \hat{K}_{32}^{(M)}(q') \int d^4q d^4k \hat{C}_2(q) \int_0^{\tau_2} d\tau_1 \int d^4p \hat{K}_{21}^{(\mu)}(k) \hat{A}_1(p)
$$
$$
\hat{\psi}_3^{(right)}(p', q') = -\lambda\Lambda \int_0^{\tau_3} d\tau_2 \hat{K}_{32}^{(m)}(p') \int d^4p d^4k \hat{A}_2(p) \int_0^{\tau_2} d\tau_1 \int d^4q \hat{K}_{21}^{(\mu)}(k) \hat{C}_1(q)
$$
(454)

In TQM, the two initial wave functions are each defined by four expectations in position, four in momentum, and four dispersions in either momentum or position space, twenty-four variables total. We have twelve integrals in momentum (or position), over the two initial particles A, C and the exchange particle B. And the two convolutions in clock time. With appropriately programmed mathematics software this is hardly a problem. But it is easy to lose sight of the physics in the course of doing the calculations.

To keep focus we will do as we have in the two previous sub-sections and fix the clock times of the vertexes as τ_X, τ_Y. Different values of τ_X, τ_Y will let us look at specific cases. The properties of the exchanged particle are key; we will focus on these.

Classical trajectories   We will take the same starting wave functions as with absorption, but with $B \to C$. A has expectations $x_a, p^{(a)}$, dispersions $\hat{\sigma}^{(a)}$; C expectations $x_c, q^{(c)}$, dispersions $\hat{s}^{(a)}$.

We assume we have A coming in from the left; C from the right; both along the x axis. Again we assume the collision is head-on with no offset along the $\hat{y}$ or $\hat{z}$ axes.

We start by fixing τ_X, τ_Y. Taking the left case first, we have for A and C:

$$x_X^{(L)} = x_a + \frac{p_x^{(a)}}{m}\tau_X$$
$$x_Y^{(L)} = x_c + \frac{q_x^{(c)}}{M}\tau_Y \tag{455}$$

And for the exchange particle:

$$x_Y^{(L)} = x_X^{(L)} + \frac{k_x^{(b)}}{\mu}\left(\tau_Y - \tau_X\right) \tag{456}$$

With $\Delta x^{(L)} \equiv x_Y^{(L)} - x_X^{(L)}, \Delta\tau \equiv \tau_Y - \tau_X$ we have the expectation of the momentum of the exchange particle:

$$k_x^{(L)} = \mu\frac{\Delta x^{(L)}}{\Delta\tau} \tag{457}$$

so the momentum of B is exactly what it needs to get from $X \to Y$ in time for its rendezvous with C. From conservation of momentum at each vertex we have for the expectations of the final momenta:

$$\langle p'_x\rangle = p_x^{(a)} - k_x^{(L)}, \langle q'_x\rangle = q_x^{(c)} + k_x^{(L)} \tag{458}$$

With the initial conditions specified and τ_X, τ_Y as well, the final expectations are immediate, with the final dispersions to be computed.

To get the right hand case we interchange the roles $A \leftrightarrow C$:

$$x_X^{(R)} = x_c + \frac{q_x^{(c)}}{M}\tau_Y$$
$$x_Y^{(R)} = x_a + \frac{p_x^{(a)}}{m}\tau_X \tag{459}$$

and:

$$k_x^{(R)} = \mu\frac{\Delta x^{(R)}}{\Delta\tau} \tag{460}$$

giving final expectations:

$$\langle p'_x\rangle = p_x^{(a)} + k_x^{(R)}, \langle q'_x\rangle = q_x^{(c)} - k_x^{(R)} \tag{461}$$

7.9.2 Exchange particle

We can write the wave function for the exchange particle in SQM as:

$$\hat{\bar{B}}_{YX}\left(\vec{k}\right) = \sqrt[4]{\frac{1}{\pi\hat{\bar{\Sigma}}^{(X)}}}e^{-i\vec{k}\cdot\vec{x}_X - \frac{1}{2}\left(\vec{k}-\vec{k}_X\right)\cdot\frac{1}{\hat{\bar{\Sigma}}^{(X)}}\cdot\left(\vec{k}-\vec{k}_X\right) - i\frac{\vec{k}^2}{2\mu}\Delta\tau} \tag{462}$$

and in TQM, first in covariant notation:

$$\hat{B}_{YX}(k) = \sqrt[4]{\frac{1}{\pi\hat{\Sigma}^{(X)}}}e^{ikx_X - \frac{1}{2}(k-k_X)\frac{1}{\Sigma^{(X)}}(k-k_X) - if_k\Delta\tau}$$
$$f_k \equiv -\frac{k^2-\mu^2}{2\mu} \tag{463}$$

and also as product of time and space parts:

$$\hat{B}_{YX}\left(k\right) = \hat{\tilde{B}}_{YX}\left(w\right) \hat{\tilde{B}}_{YX}\left(\vec{k}\right) \exp\left(\imath \frac{\mu}{2}\Delta\tau\right) \tag{464}$$

$$\hat{\tilde{B}}_{YX}\left(w\right) = \sqrt[4]{\frac{1}{\pi\hat{\tilde{\sigma}}^{(X)}}}\, e^{\imath w t_X - \frac{1}{2}\frac{(w-w_X)^2}{2\hat{\tilde{\sigma}}^{(X)}} + \imath \frac{w^2}{2\mu}\Delta\tau} \tag{465}$$

Taking the left side for definiteness, the wave function of the intermediate state is properly the direct product of the highly correlated wave functions of A, B with the (as yet) uncorrelated wave function of the C particle. After B encounters C at Y, B is gone and now A and C are highly correlated.

We have proceeded a bit formally, specifying τ_X, τ_Y and then deriving the properties of the exchange particle from these. A more physical approach might be to fix the momentum of the exchange particle and then integrate over all values of τ_X (or τ_Y) consistent with that.

7.9.3 Discussion

In any case, if we are interested in the use of one particle as a measurement of the other, then we already have what we need: B inherits its dispersions from its parent. If its parent was narrow in time/space, then it will be as well and act as a de facto gate with respect to the other particle.

If we are interested in doing a Bell's theorem correlation in time, then we need to track the correlations through. Because momentum is conserved at each vertex, the outgoing particles will be highly correlated in momentum space.

If we are interested in symmetry properties in the time direction, then we can:

1. Take our C as really an A,

2. Start with wave functions that are symmetric under particle exchange but which nevertheless have a component which is anti-symmetric in time (as the wide and narrow wave function in section 7.3),

3. And then see what breaks.

In SQM we expect that we will see scattering which is symmetric in space at each clock tick. In TQM we expect that we will see a component which is anti-symmetric at specific clock ticks, but which still preserves complete symmetry when both space and time are interchanged.

If we are interested in the bound case, we can look at the exchange particle as creating a Yukawa force in time. In this case we will need to track the factors of $\exp\left(-\imath\mu\tau\right)$. Note that the $\exp\left(-\imath f_k\tau\right)$ factor will tend to keep the exchange particle on-shell for larger values of $\Delta\tau$. In coordinate space this will tend to make the effective potential look like a Liénard–Wiechert potential.
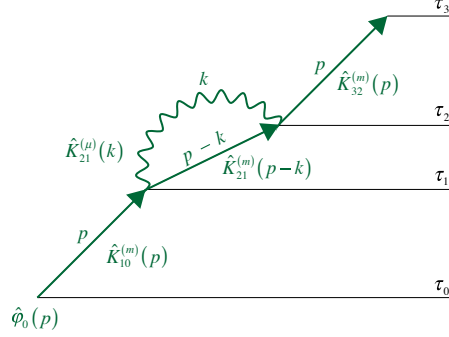
Figure 17: Loop correction to the mass

7.10 Loop correction to the mass

How do we calculate loop diagrams in TQM?

7.10.1 Loop correction in SQM

In the ABC model there is an amplitude λ for an A to emit a B then absorb it. As is well known, in quantum field theory this can be made to look like a correction to the mass, with the effect of taking a bare mass to a corrected mass: $m_0^2 \rightarrow m^2 = m_0^2 + \delta m^2$. Unfortunately the integral for this δm^2 correction is divergent. If the A particle has four-momentum p and emits a B with four-momentum k, to compute the amplitude associated with the loop we will need to integrate over all possible values of the intermediate k:

$$\delta m^2 \sim \int d^4 k \frac{\imath}{(p-k)^2 - m^2} \frac{\imath}{k^2 - \mu^2} \qquad (466)$$

This is logarithmically divergent at large k:

$$\delta m^2 \sim \int \frac{d^4 k}{k^4} \qquad (467)$$

As a result all such integrals have to be regularized: a convergence factor has to be inserted which acts as an effective cutoff, throwing out the high energy part of the loop in a way that does not distort results at lower energies.

To do this we take advantage of the general principle that all physical measurements involve an implicit comparison between two measurements. If we are going to use the mass of a particle as a value in one calculation we must first have found that mass in another. An absolute, standalone measurement is not possible even in principle. All measurements have to be renormalized–normalized by comparison to another – to get physically meaningful numbers.

In quantum field theory renormalization – needed physically – is also used to regularize, to contain and control the infinities. If we make the necessary comparison in the right way, we can use it to subtract off the infinities.

It's a bit like weighing a mouse by first weighing an aircraft carrier without the mouse, weighing the aircraft carrier with the mouse on board, and then subtracting out the weight of the aircraft carrier to get the weight of the mouse.

There is no guarantee that this will work. What if the cutoff function/procedure being used on both sides of the comparison has unintended side-effects at lower energies? But in spite of the obvious risk the procedure works — and brilliantly – producing some of the most accurate predictions in the whole of physics.

7.10.2 Loop correction in TQM

With an extra dimension to integrate over we might reasonably expect that the corresponding loop integrals in TQM would not only be divergent, but perhaps even be divergent in a way which cannot be contained by renormalization.

We will first look at the effects of a simple mass correction term $\frac{\delta m^2}{2m_0}$, then at the full loop calculation.

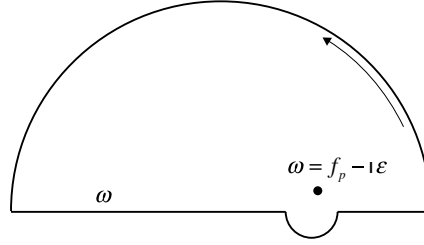### 7.10.3 Simple mass correction

Figure 18: Contour diagram for free propagator

We are first going to look at a simple mass correction. We start with a "bare" mass, then add a correction term:

$$m_0^2 \rightarrow m_0^2 + \delta m^2 \tag{468}$$

Kernel from above:

$$\left(i \frac{\partial}{\partial \tau} + \frac{p^2 - m_0^2}{2m_0} \right) K_\tau^{(0)}(x; x') = \delta(\tau) \delta^4(x - x') \tag{469}$$

In momentum space:

$$\left(i \frac{\partial}{\partial \tau} - f_p \right) \hat{K}_\tau^{(0)} = \delta(\tau) \tag{470}$$

giving:

$$\hat{K}_\tau^{(0)} = \exp(-i f_p \tau) \theta(\tau) \tag{471}$$

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with the usual $f_p \equiv -\frac{p^2 - m_0^2}{2m_0}$.

We write the Schrödinger equation as:

$$\imath \frac{\partial}{\partial \tau} = -\frac{p^2 - m_0^2 - \delta m^2}{2m_0} = H_0 + V \tag{472}$$

with:

$$H_0 = -\frac{p^2 - m_0^2}{2m_0}, V = \frac{\delta m^2}{2m_0} \tag{473}$$

The power series solution is a series of convolutions:

$$\hat{K}_T = \hat{K}_T^{(0)} - \imath \frac{\delta m^2}{2m_0} \int_0^T d\tau_1 \hat{K}_{T1}^{(0)} \hat{K}_1^{(0)} - \frac{\delta m^4}{4m_0^2} \int_0^T d\tau_2 \hat{K}_{T2}^{(0)} \int_0^{\tau_2} d\tau_1 \hat{K}_{21}^{(0)}(p) \hat{K}_1^{(0)} + \ldots \tag{474}$$

or more compactly:

$$\hat{K} = \hat{K}_0 - \imath \frac{\delta m^2}{2m_0} \hat{K}_0 * \hat{K}_0 - \frac{\delta m^4}{4m_0^2} \hat{K}_0 * \hat{K}_0 * \hat{K}_0 + \ldots \tag{475}$$

As this is a series of convolutions we can use Fourier transforms with respect to clock time to solve.

To avoid having to write expressions like $\hat{\hat{K}}$ for the Fourier transform (with respect to τ) of something which is already a Fourier transform (with respect to x) we define $g_p \equiv \hat{K}_\tau^{(0)}(p), G_p \equiv \hat{K}_\tau(p)$.

We want g_p in terms of its Fourier transform:

$$g_p(\tau) = \frac{1}{\sqrt{2\pi}} \int d\omega \exp(-\imath \omega \tau) \hat{g}_p(\omega) \tag{476}$$

We will use contour analysis to compute this. Since we are only interested in positive times, we close the contour in ω above the real line:

$$\tau > 0 \Rightarrow \exp(-\imath(-\imath\varepsilon)\tau) \rightarrow \exp(-\varepsilon\tau) \tag{477}$$

With this condition the correct propagator in ω space is:

$$\hat{g}_p(\omega) = -\frac{\imath}{\sqrt{2\pi}} \frac{1}{\omega - f_p + \imath\varepsilon} \tag{478}$$

We therefore have:

$$G_p = g_p - \imath \frac{\delta m^2}{m_0} g_p * g_p - \frac{\delta m^4}{m_0^2} g_p * g_p * g_p + \ldots \tag{479}$$

$$\hat{G}_p = \hat{g}_p - \sqrt{2\pi}\imath \frac{\delta m^2}{2m_0} \hat{g}_p^2 - \left(\sqrt{2\pi}^{-2} \frac{\delta m^2}{2m_0}\right)^2 \hat{g}_p^3 + \ldots \tag{480}$$

We have a geometric series so can sum:

$$\hat{G}_p = \frac{\hat{g}}{1 + \sqrt{2\pi}\imath\frac{\delta m^2}{2m_0}\hat{g}} \tag{481}$$

expanding $\hat{g}$:

$$\hat{G}_p = -\frac{\imath}{\sqrt{2\pi}}\frac{1}{\omega - \frac{p^2 - m_0^2 - \delta m^2}{2m_0} + \imath\varepsilon} \tag{482}$$

and inverting get the obviously correct:

$$G_p = \exp\left(\imath\frac{p^2 - m_0^2 - \delta m^2}{2m_0}\tau\right)\theta(\tau) \tag{483}$$

7.10.4 Loop mass correction

We now look at the loop correction. We get the same kind of series as with the simple mass correction.

$$\hat{K} = \hat{K}^{(0)} - \lambda^2\hat{K}^{(0)} * L(p) * \hat{K}^{(0)} + \lambda^4\hat{K}^{(0)} * L(p) * \hat{K}^{(0)} * L(p) * \hat{K}^{(0)} - \ldots \tag{484}$$

with the loop integral:

$$L_\tau(p) \equiv \int dk\hat{K}_\tau(p - k)\hat{K}_\tau^{(\mu)}(p - k) \tag{485}$$

Written as convolutions in clock time we have:

$$\hat{G}_p = \hat{g}_p - 2\pi\lambda^2\hat{g}_p\hat{L}_\omega(p)\hat{g}_p + 4\pi^2\lambda^4\hat{g}_p\hat{L}_\omega(p)\hat{g}_p\hat{L}_\omega(p)\hat{g}_p - \ldots \tag{486}$$

Where $\hat{L}_\omega(p)$ is the Fourier transform with respect to clock time of the loop integral.

We again have a geometric series so we get:

$$\hat{G}_p = \hat{g}_p\frac{1}{1 + 2\pi\lambda^2\hat{g}_p\hat{L}_\omega(p)} \tag{487}$$

To compute this we break the problem down into two steps: computing $L_\tau(p), \hat{L}_\omega(p)$.

Loop integral for fixed clock time We apply the loop integral to a initial Gaussian test function. In momentum space:

$$L_\tau(p) = \int d^4k\hat{K}_\tau^{(m)}(p - k)\hat{K}_\tau^{(\mu)}(k)\hat{\varphi}_0(p) \tag{488}$$

with initial Gaussian test function:

$$\hat{\varphi}_0\left(p\right) = \sqrt[4]{\frac{1}{\pi^4 \det\left(\hat{\Sigma}\right)}} \exp\left(-\frac{\left(p - p_0\right)^2}{2\Sigma}\right) \tag{489}$$

As with the case of particle absorption, we note that the integral over the internal momentum is a convolution in momentum space, so it is a product in coordinate space. The corresponding loop integral in coordinate space is:

$$L_\tau\left(x_1\right) = 4\pi^2 \int d^4 x_0 K_\tau^{(m)}\left(x_1; x_0\right) K_\tau^{(\mu)}\left(x_1; x_0\right) \varphi_0\left(x_0\right) \tag{490}$$

The kernels in coordinate space are:

$$\begin{aligned}
K_\tau^{(m)}\left(x_1; x_0\right) &= -\imath\frac{m^2}{4\pi^2\tau^2} \exp\left(-\frac{\imath m}{2\tau}\left(x_1 - x_0\right)^2 - \imath\frac{m}{2}\tau\right) \\
K_\tau^{(\mu)}\left(x_1; x_0\right) &= -\imath\frac{\mu^2}{4\pi^2\tau^2} \exp\left(-\frac{\imath\mu}{2\tau}\left(x_1 - x_0\right)^2 - \imath\frac{\mu}{2}\tau\right)
\end{aligned} \tag{491}$$

The product equals a single coordinate space kernel:

$$K_\tau^{(M)}\left(x_1; x_0\right) = -\imath\frac{M^2}{4\pi^2\tau^2} \exp\left(-\imath\frac{M}{2\tau}\left(x_1 - x_0\right)^2 - \imath\frac{M}{2}\tau\right) \tag{492}$$

with a modified mass $M \equiv m + \mu$ and a prefactor.
So the loop integral in coordinate space is now:

$$L_\tau\left(x_1\right) = -\imath\frac{m^2\mu^2}{\tau^2 M^2} \int d^4 x_0 K_\tau^{(M)}\left(x_1; x_0\right) \varphi_0\left(x_0\right) \tag{493}$$

The presence of the Gaussian test function on the right means we get by inspection:

$$L\tau\left(x_1\right) = -\imath\frac{m^2\mu^2}{\tau^2 M^2} \varphi_\tau^{(M)}\left(x_1\right) \tag{494}$$

We have a correction that shows a spread in time, but at the slightly slower rate associated with the slightly larger mass M. Further the correction is much greater at shorter clock times.

$$\begin{aligned}
L_\tau\left(p_1\right) &= -\imath\frac{m^2\mu^2}{\tau^2 M^2} \int d^4 p_0 \hat{K}_\tau^{(M)}\left(p_1; p_0\right) \hat{\varphi}_0\left(p_0\right) \\
\hat{K}_\tau^{(M)}\left(p_1; p_0\right) &= \exp\left(\imath\frac{p_0^2 - m^2}{2m}\tau\right) \delta^4\left(p_1 - p_0\right)
\end{aligned} \tag{495}$$

So the loop correction for fixed clock time is:

$$L_\tau\left(p\right) = -\imath\frac{m^2\mu^2}{M^2\tau^2} \exp\left(\imath\frac{p^2 - M^2}{2M}\tau\right) \hat{\varphi}_0\left(p\right) \tag{496}$$

At this point the value of the loop correction at a particular value of p is independent of the specific shape of the incoming wave function. We are therefore free to drop the initial wave function from the analysis:

$$L_\tau(p) = -\imath \frac{m^2 \mu^2}{M^2 \tau^2} \exp\left(\imath \frac{p^2 - M^2}{2M} \tau\right) \tag{497}$$

We have selected this specific case as our demonstration partly because we can neglect the incoming wave function once we have the loop correction at a specific p. This is not necessarily the case in general.

As we have seen throughout, TQM implies the initial wave functions:

1. have finite dispersion in time (as well as space)

2. are entangled with the loop integration variables

This in turn means that each loop integration picks up a Gaussian factor that guarantees its convergence, and then passes a Gaussian factor (usually a bit wider) along to the next step in the calculation.

The combination of finite initial dispersion and entanglement forces convergence.

Note this approach will not work in SQM. In SQM there is by assumption no finite initial dispersion in time. And even if there were, each step is cut off from the previous since there is no entanglement in time.

Fourier transform of the loop integral over clock time Now that we have the loop integral for a specific value of the clock time we can take the Fourier transform with respect to τ:

$$\hat{L}_\omega(p) = -\imath \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau \frac{m^2 \mu^2}{M^2 \tau^2} e^{\imath(\omega - F_p)\tau} \tag{498}$$

We define $F_p \equiv -\frac{p^2 - M^2}{2M}$. For small μ, $F_p \approx f_p$.

We use contour integration to compute this. We will use Feynman boundary conditions rather than retarded because the results are slightly simpler if we do so.

We mean by this that when $\omega > f_P$ we go in the positive clock time direction; when $\omega < f_P$ we go in the negative clock time direction. (This is the first and only time we allow movement in the negative clock time direction.) In both cases we will take the pole at $\tau = 0 + \imath\varepsilon$ and close the contour above.

We will first look at the case for $\omega - F_p > 0$. We have a pole of second order. The residue is:

$$Res(h) = \lim_{\varepsilon \to 0} \frac{d}{d\tau} h(\tau) \tag{499}$$

with:

$$\begin{aligned} h(\tau) &= -\imath \frac{1}{\sqrt{2\pi}} \frac{m^2 \mu^2}{M^2} \exp\left(\imath (\omega - F_P) \tau\right) \\ Res(h) &= \frac{1}{\sqrt{2\pi}} \frac{m^2 \mu^2}{M^2} (\omega - F_p) \end{aligned} \tag{500}$$

or:

$$\hat{L}_\omega(p) = \sqrt{2\pi}\imath\frac{m^2\mu^2}{M^2}(\omega - F_P) \tag{501}$$

With Feynman boundary conditions the case $\omega - F_p < 0$ is identical. (With retarded boundary conditions we would have had to $(\omega - F_P) \rightarrow |\omega - F_P|$, which would in turn have created a second pole in the inversion from clock frequency to clock time below. The results would be more complex, but there is no problem of principle.)

Substituting back in the original expression (equation 487):

$$\hat{G}_p = -\frac{\imath}{\sqrt{2\pi}}\frac{1}{(1+a)\,\omega + (f_p + af_P) + \imath\varepsilon}, a \equiv 2\pi\lambda^2\mu^2 \tag{502}$$

This has a pole at:

$$\omega = f'_p = \frac{f_p + af_P}{1+a} = \frac{1}{1+a}\left(-\frac{p^2 - m^2}{2m} - a\frac{p^2 - (m+\mu)^2}{2(m+\mu)}\right) \tag{503}$$

giving for the corrected propagator:

$$\exp(-\imath f_p\tau) \rightarrow \frac{1}{1+a}\exp(-\imath f'_p\tau) \tag{504}$$

If the loop correction had depended on the specifics of the initial Gaussian test function, we would only have been able to supply the correction from the first loop term. We would not have been able to sum the infinite series of loop corrections in one expression.

We still need to renormalize. We could take the value of f'_p on-shell as the starting point, then examine the behavior as we move further off-shell.

But we have separated the problems associated with renormalization from those associated with regularization.

7.10.5 Discussion of the loop corrections

At this point we have established that TQM does not need to be regularized.

We have only looked at a toy case. But the principles established here apply generally. As noted, the combination of Morlet wavelet analysis and entanglement in time mean that the integrals encountered in a diagram are self-regularized: the Gaussian functions which are passed through a series of integrals easily dominate any polynomial divergence.

An implication is that it is the assumption that quantum mechanics does not apply along the time dimension that is responsible for the ultraviolet divergences. The familiar divergences are a side-effect of not pushing the ideas of quantum mechanics and special relativity hard enough, of our failure to treat time and space symmetrically in quantum mechanics.

7.11 Discussion

We have established that we can extend TQM to include the multiple particle case. Equivalently, that we can extend field theory to include time on the same basis as space.

TQM is conceptually simpler than SQM: time and space are treated on an equal footing and there are no ultraviolet divergences. But it is calculationally more complex: we have coordinate time to consider and we are required to use Gaussian test functions rather than plane waves as the fundamental unit of analysis.

We have examined the basic parts of a Feynman diagram: free propagators; the emission, absorption, and exchange of a particle; and simple loop diagrams. We are therefore able to work out – in principle at least – the results for any diagram in a perturbation expansion. And therefore to compare TQM to SQM in any experiment which can be described by such expansions.

To falsify TQM we need experiments that work at short times and with individual wave packets. Long times and averages over wave packets kill the effects associated with dispersion in time.

The most decisive such experiments are likely to be ones that emphasize the effects of the time/energy uncertainty principle.

Additional effects include:

1. anti-symmetry in time,

2. correlations in time (Bell's theorem in time),

3. forces of anticipation and regret.

To be sure, these latter effects likely to be both subtle and small. Therefore they may not be that useful for falsifiability, our primary target in this work. But they are interesting in their own right.

8   Discussion

> "It is difficult to see what one does not expect to see." – William FellerFeller [1968]

8.1   Falsifiability

With the single slit in time we have a decisive test of temporal quantum mechanics. In SQM, the narrower the slit, the *less* the dispersion in subsequent time-of-arrival measurements. In TQM, the narrower the slit, the *greater* the subsequent dispersion in subsequent time-of-arrival measurements. *In principle, the difference may be made arbitrarily great.*

To get to this point we had to develops the rules in a way that does not admit of free parameters or other ways to significantly modify the predictions. To do this:

1. We took path integrals as the defining representation. This made the extension from three to four dimensions unambiguous.

2. We used Morlet wavelet analysis rather than Fourier analysis to define the initial wave functions. This let us avoid the use of unphysical wave functions and made achieving convergence and normalization of the path integrals possible.

3. We distinguished carefully between "clock time" and "coordinate time".

As a result, once we have applied the requirement that TQM match SQM in the appropriate limit, there are no free parameters.

That does not of itself prove that there is not another way to apply quantum mechanics along the time dimension. For instance, one could start from a Hamiltonian approach, see for instance Yau [2015]. We have therefore been careful to focus on dimensional and symmetry arguments, which give first order predictions which are likely to be independent of the specifics of whatever method we might use. These first order predictions of TQM do not have much "give":

- The initial dispersion in time is fixed by symmetry between time and space and the principle of maximum entropy.

- The evolution of the wave functions is fixed by the long, slow approximation. This allows for give, but only over times of picoseconds, glacial by the standards of TQM.

- We could choose Alice's frame or Bob's to do the analysis, but the corrections due to this are of second order. The corrections can be eliminated entirely by selecting "the rest frame of the vacuum" as the defining frame for TQM.

The predictions of TQM are therefore falsifiable to first order.

8.2   Experimental effects

We have discussed two primary effects:

1. generally increased dispersion in time, as time-of-arrival effects.

2. the time/energy uncertainty principle, as the single slit in time.

These effects should be present in any experimental setup in which the sources vary in time and the detectors are time-sensitive.

Additionally we can look for:

- Forces of anticipation and regret. As the paths in TQM advance into the future they will encounter potentials earlier (anticipation) than in SQM. And as they dive back into the past they will continue to interact with potentials later (regret) than is the case in SQM.

- Shadowing in time – self-interference by detectors and sources.

- Correlations in time (Bell's theorem in time). Particles that have interacted in the past, as in EPR experiments, will be entangled in time as well as space.

- Anti-symmetry in time. Wave functions are free to satisfy their symmetry requirements using the time dimension as well as the three space dimensions.

In general, any quantum effect seen in space is likely have an "in time" variation. TQM is to SQM with respect to time as SQM is to classical mechanics with respect to space.

Reviews of foundational experiments in quantum mechanics (for example Lamoreaux Lamoreaux [1992], Ghose Ghose [1999], and Auletta Auletta [2000]) provide a rich source of candidate experiments: the single and double slit as well as many other foundational experiments have an "in time" variant, typically with time and a space dimension flipped.

The experiments are likely to be difficult. The attosecond times we are primarily interested in are at the edge of the detectible. The investigation here was partly inspired by Lindner's "Attosecond Double-Slit Experiment" Lindner et al. [2005]. But the times there, $500as$, are far too long for us. More recent work has reached shorter and shorter times: 12 attoseconds in Koke, Sebastian, and Grebing Koke et al. [2010] and as noted the extraordinary sub-attosecond times in Ossiander Ossiander et al. [2016]. The effects of dispersion in time should now be within experimental range.

8.3   Further extensions

We have provided only a basic toolkit for TQM. Areas for further investigation include:

1. Generalizing the treatment of spinless massive bosons to include photons and fermions. Extension to the Standard Model.

2. Derivation of the bound state wave functions from first principles.

3. More detailed treatment of scattering experiments.

4. Exact treatment of the single slit in time, included paths that wander back and forth through the slit.

5. Careful treatment of measurements, including paths that overshoot, undershoot, and loop around the detector.

6. Decoherence in time.

7. Infrared divergences. From the point of view of TQM, these may be the flip side of the ultraviolet divergences, suppressed in a similar way.

8. More detailed treatment of the ultraviolet divergences.

9. Examination of the spin-statistics connection: the T in the CPT theorem must come under grave suspicion.

10. Statistical mechanics. Any statistical ensemble should include fluctuations in time. Implicit in TQM is the possibility that the initial smooth wave function of wave mechanics should itself be replaced by a statistical ensemble of fluctuations in time.

11. More detailed treatment of the choice of frame.

12. Quantum gravity. Since TQM is by construction highly symmetric between time and space and free of the ultraviolet divergences, it may be a useful starting point for attacks on the problem of quantum gravity.

8.4 Five requirements

In their delightfully titled *How to Think about Weird Things*Schick Jr. and Vaughan [1995] the philosophers Schick and Vaughn lay out five requirements that a hypothesis such as TQM should satisfy:

1. **Testability** – *are there experimental tests? ideally: is the hypothesis falsifiable?* TQM has no free parameters; it can therefore be falsified by any experiment at appropriate scale looking at time varying quantum phenomena.

2. **Fruitfulness** – *does the hypothesis suggest new lines of research, new phenomena to explore?* All time-varying quantum phenomena offers targets for investigation. The list of experimental effects given above is doubtless far from exhaustive.

3. **Scope** – *how widespread are the phenomena?* TQM applies to all time-varying quantum phenomena.

4. **Simplicity** – *does it make the fewest possible assumptions?* TQM eliminates the assumption that time and space should be treated differently in quantum mechanics. It also eliminates the ultraviolet divergences and the consequent need to regularize the loop integrals in field theories (as QED).

5. **Conservatism** – *is it consistent with what is known?* TQM matches SQM in the long time (picosecond) limit.

8.5 No null experiments

This concludes the argument for TQM.

Now suppose that one or more of the proposed experiments is done and conclusively demonstrates that TQM is false. That would in turn raise some interesting questions:

1. Is there a frame in which TQM is maximally (or minimally) falsified? That would be a preferred frame, anathema to relativity.

2. Is TQM equally false in all frames? if it is false in all frames, how do we reconcile the disparate wave functions of Alice and Bob?

As TQM is a straight-forward extrapolation of quantum mechanics and special relativity, experiments that falsify TQM are likely to require modification of our understanding of either quantum mechanics or special relativity or both. There are no null experiments.

Acknowledgments

A Conventions

We use natural units $\hbar = c = 1$. When summing over three dimensions we use i, j, k. We use $\imath$ sans dot for the square root of -1; i with dot for the index variable. When summing over four dimensions we use h for the coordinate time index so such sums run over h, i, j, k.

A.1 Clock time

We use τ for clock time. The use of the Greek letter τ for clock time is meant to suggest that this is a "classical" time. We use f for its complementary variable, clock frequency:

$$f^{(op)} \equiv \imath \frac{\partial}{\partial \tau} \tag{505}$$

The clock time τ will usually be found at the bottom right of any symbol it is indexing:

$$\varphi_\tau, K_\tau \tag{506}$$

The clock time is in its turn frequently indexed: $\tau_0, \tau_1, \tau_2, \ldots$

As a result, deeply nested subscripts are an occasional hazard of this analysis. To reduce the nesting level we use obvious shortenings, i.e.:

$$\varphi_{\tau_1}(x_1) \rightarrow \varphi_1(x_1) \rightarrow \varphi_1 \tag{507}$$

$$K_{\tau_1 \tau_0}(x_1; x_0) \rightarrow K_{10}(x_1; x_0) \rightarrow K_1(x_1; x_0) \rightarrow K_1 \tag{508}$$

And we represent differences in clock time by combining indexes:

$$\tau_{21} \equiv \tau_2 - \tau_1 \tag{509}$$

A.2 Coordinate time and space

We use $E, \vec{p}$ for the momentum variables complementary to coordinate time t and space $\vec{x}$:

$$E^{(op)} \equiv \imath \frac{\partial}{\partial t}$$
$$\vec{p}^{(op)} \equiv -\imath \nabla \tag{510}$$

When there is a natural split into coordinate time and space parts we use a tilde to mark the time part, an overbar to mark the space part. For example:

$$\psi(t, \vec{x}) = \tilde{\psi}_\tau(t) \bar{\psi}_\tau(\vec{x}) \tag{511}$$

This is to reinforce the idea that in this analysis the three dimensional part is the average (hence overbar), while the coordinate time part contributes a

bit of quantum fuzziness (hence tilde) on top of that. With that said, we will sometimes omit the overbar and the tilde when they are obvious from context:

$$\tilde{\varphi}_\tau (t) \bar{\varphi}_\tau (\vec{x}) \rightarrow \varphi_\tau (t) \varphi_\tau (\vec{x}) \tag{512}$$

We use an overdot to indicate the partial derivative with respect to laboratory time:

$$\dot{g}_\tau (t, \vec{x}) \equiv \frac{\partial g_\tau (t, \vec{x})}{\partial \tau} \tag{513}$$

A.3   Fourier transforms

We use a caret to indicate that a function or variable is being taken in momentum space. To keep the Fourier transform itself covariant we use opposite signs for the coordinate time and space parts:

$$
\begin{aligned}
\hat{g} (E, \vec{p}) &= \tfrac{1}{\sqrt{2\pi}^4} \int\limits_{-\infty}^{\infty} dt d\vec{x} \, e^{\imath E t - \imath \vec{p} \cdot \vec{x}} g (t, \vec{x}) \\
g (t, \vec{x}) &= \tfrac{1}{\sqrt{2\pi}^4} \int\limits_{-\infty}^{\infty} dE d\vec{p} \, e^{-\imath E t + \imath \vec{p} \cdot \vec{x}} \hat{g} (E, \vec{p})
\end{aligned}
\tag{514}
$$

For plane waves:

$$
\begin{aligned}
\phi_p (x) &= \tilde{\phi} (t) \bar{\phi} (\vec{x}) = \tfrac{1}{\sqrt{2\pi}} \exp (-\imath E t) \tfrac{1}{\sqrt{2\pi}^3} \exp (\imath \vec{p} \cdot \vec{x}) \\
\hat{\phi}_x (p) &= \hat{\tilde{\phi}} (E) \hat{\bar{\phi}} (\vec{p}) = \tfrac{1}{\sqrt{2\pi}} \exp (\imath E t) \tfrac{1}{\sqrt{2\pi}^3} \exp (-\imath \vec{p} \cdot \vec{x})
\end{aligned}
\tag{515}
$$

To shorten the expressions we use:

$$x \equiv (t, \vec{x}) = (t, x, y, z) \tag{516}$$

The difference between x the four vector and x the first space coordinate should be clear from context. In momentum space we use:

$$p \equiv (E, \vec{p}) = (E, p_x, p_y, p_z) \tag{517}$$

We have similar rules for clock time τ and its complementary energy f:

$$
\begin{aligned}
\hat{g}_f &= \tfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} d\tau \, e^{\imath f \tau} g_\tau \\
g_\tau &= \tfrac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} df \, e^{-\imath f \tau} \hat{g}_f
\end{aligned}
\tag{518}
$$

With these conventions when a Fourier transform is given by a convolution:

$$\hat{h} (p) = \int dk \hat{f} (p - k) \hat{g} (k) \tag{519}$$

the function in coordinate space is given by:

$$h(x) = \sqrt{2\pi} f(x) g(x) \tag{520}$$

When it is obvious that a symbol represents a Fourier transform we may drop the caret:

$$\hat{\varphi}(p) \rightarrow \varphi(p) \tag{521}$$

A.3.1 Gaussian test functions

As Gaussian test functions play a critical role in this investigation it is useful to have a consistent notation with which to describe them.

In general we define a Gaussian test function as a normalized Gaussian function. It may be in position or momentum space. It is defined by its expectations for position and momentum and by its dispersion in either position or momentum. The most important single example here is the Gaussian test function that describes the time part of the wave function of a free particle:

$$\tilde{\varphi}_\tau(t) = F_\tau^{(t)} e^{-\imath E_0 t - \imath \frac{E_0^2}{2m} \tau - \frac{1}{2\sigma_t^2 f_\tau^{(t)}} \left(t - t_0 - \frac{E_0}{m}\tau\right)^2} \tag{522}$$

with dispersion factor:

$$f_\tau^{(t)} \equiv 1 - \imath \frac{\tau}{m\sigma_t^2} \tag{523}$$

and normalization factor:

$$F_\tau^{(t)} \equiv \sqrt[4]{\frac{1}{\pi\sigma_t^2}} \sqrt{\frac{1}{f_\tau^{(t)}}} \tag{524}$$

The sign of the complex part of a dispersion factor is negative for time, positive for space:

$$f_\tau^{(x)} \equiv 1 + \imath \frac{\tau}{m\sigma_x^2} \tag{525}$$

The "complex dispersion" is:

$$\sigma_t^2 f_\tau^{(t)} = \sigma_t^2 - \imath \frac{\tau}{m} \tag{526}$$

So the absolute value of the complex dispersion or absolute dispersion is:

$$\sigma_\tau^{(t)2} \equiv \sqrt{\sigma_t^2 + \frac{\tau^2}{m^2}} \tag{527}$$

If we are dealing with multiple Gaussian test functions we may make name changes $f \rightarrow g, F \rightarrow G,\ f \rightarrow h, F \rightarrow H$ to the dispersion and normalization factors.

The expectation and dispersion of coordinate time t are given by:

$$\langle t \rangle = \int dt t |\tilde{\varphi}_\tau(t)|^2 = t_0$$
$$(\Delta t)^2 = \langle t^2 \rangle - \langle t \rangle^2 = \frac{\sigma_t^2}{2} \tag{528}$$
$$\langle t^2 \rangle = \int dt t^2 |\tilde{\varphi}_\tau(t)|^2$$

The expressions for coordinate energy E are given by the Fourier transforms of the expression for coordinate time t, e.g.:

$$\hat{\tilde{\varphi}}_0(E) \equiv \sqrt[4]{\frac{1}{\pi \sigma_E^2}} e^{\imath E t_0 - \frac{(E - E_0)^2}{2\sigma_E^2}} \tag{529}$$

With expectation and dispersion of coordinate energy E:

$$\langle E \rangle = \int dE E \left| \hat{\tilde{\varphi}}_0(E) \right|^2 = E_0$$
$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 = \frac{\sigma_E^2}{2} \tag{530}$$
$$\langle E^2 \rangle = \int dE E^2 \left| \hat{\tilde{\varphi}}_0(E) \right|^2$$

In general we can switch between time/energy and space/momentum forms by taking the complex conjugate and interchanging variables $t \leftrightarrow x$. This is our own small version of the CPT transformations.

As expressions like $\sigma_{p_x}^2$ are cumbersome we sometimes replace them with $\hat{\sigma}_x^2 \equiv \sigma_{p_x}^2$ taking implicit advantage of the fact that with our conventions $\sigma_x^2 = \frac{1}{\sigma_{p_x}^2}$.

If we need to tag various Gaussian wave functions we may assign each specific letter a, b, c as:

$$\tilde{\varphi}_\tau^{(a)}(t) = F_\tau^{(a)} e^{-\imath E_a t - \imath \frac{E_a^2}{2m} \tau - \frac{1}{2\sigma_t^{(a)2} f_\tau^{(a)}} \left(t - t_a - \frac{E_a}{m} \tau \right)^2} \tag{531}$$

We usually use φ for Gaussian test functions but may use a capital letter to reduce notational clutter, as $A \equiv \varphi^{(a)}$.

In this text Gaussian test functions in three or four dimensions are always simple products of single Gaussian test functions:

$$\hat{\tilde{\varphi}}_\tau(\vec{p}) = \hat{\varphi}_\tau(p_x) \hat{\varphi}_\tau(p_y) \hat{\varphi}_\tau(p_z) \tag{532}$$

or:

$$\hat{\varphi}_\tau(p) = \hat{\varphi}_\tau(E) \hat{\varphi}_\tau(p_x) \hat{\varphi}_\tau(p_y) \hat{\varphi}_\tau(p_z) \tag{533}$$

A.4 Acronyms

CM Classical Mechanics: all four dimensions treated as parameters.

SQM Standard Quantum Mechanics: quantum mechanics with the three space dimensions treated as observables, time as a parameter.

TQM Temporal Quantum Mechanics: SQM but with time treated as an observable on the same basis as the three space dimensions.

B   Classical equations of motion

We verify that we get the classical equations of motion from the Lagrangian.
Broken out into time and space parts the Lagrangian is:

$$L\left(t, \vec{x}, \dot{t}, \dot{\vec{x}}\right) = -\frac{1}{2}m\dot{t}^2 + \frac{1}{2}m\dot{\vec{x}} \cdot \dot{\vec{x}} - q\dot{t}\Phi\left(t, \vec{x}\right) + q\dot{x}_j A_j\left(t, \vec{x}\right) - \frac{1}{2}m \tag{534}$$

The Euler-Lagrange equations are:

$$\frac{d}{d\tau}\frac{\delta L}{\delta \dot{x}^\mu} - \frac{\delta L}{\delta x^\mu} = 0 \tag{535}$$

From the Euler-Lagrange equations we have:

$$m\ddot{t} = -q\dot{\Phi} + q\dot{t}\Phi_{,0} - q\dot{x}_j A_{j,0} = -q\dot{x}_j\left(\Phi_{,j} + A_{j,0}\right) \tag{536}$$

$$m\ddot{x}_i = -q\dot{A}_i - qt\Phi_{,i} + q\dot{x}_j A_{j,i} = -q\dot{t}A_{i,0} - q\dot{x}_j A_{i,j} - q\Phi_{,i}\dot{t} + q\dot{x}_j A_{j,i} \tag{537}$$

Here the Roman indexes, i and j, go from 1 to 3 and if present in pairs are summed over. We use an overdot to indicate differentiation by the laboratory time τ.

By using:

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{538}$$

$$\vec{B} = \nabla \times \vec{A} \tag{539}$$

we get:

$$m\ddot{t} = q\vec{E} \cdot \dot{\vec{x}} \tag{540}$$

and:

$$m\ddot{\vec{x}} = q\dot{t}\vec{E} + q\dot{\vec{x}} \times \vec{B} \tag{541}$$

which are the familiar equations of motion of a classical particle in an electromagnetic field if we take τ as the proper time of the particle.

C   Ehrenfest theorem

Ehrenfest's theorem provides a natural connection between quantum and classical mechanics. It shows that the expectation values of quantum mechanical operators obey the classical equations. It demonstrates the consistency of SQM with classical mechanics in the sense that the expectations in SQM obey the classical equations of motion. We now derive it for TQM, adapting the standard proofs Potter and Messiah [1958], Baym [1969], Liboff [1997], Merzbacher [1998] to TQM.

The Hamiltonian acts as the generator of translations in laboratory time:

$$\imath \frac{\partial}{\partial \tau} \psi = H \psi \tag{542}$$

With H:

$$H = -\frac{(p - qA)^2 - m^2}{2m} = -\frac{1}{2m} \left((\imath \partial_\mu - qA_\mu)(\imath \partial^\mu - qA^\mu) - m^2 \right) \tag{543}$$

We use this to compute the change of expectation values of an observable operator O with laboratory time. The derivative of the expectation is given by:

$$\frac{d \langle O \rangle}{d\tau} = \frac{\langle O \rangle_{\tau + \varepsilon} - \langle O \rangle_\tau}{\varepsilon} = \frac{1}{\varepsilon} \left(\int dq \, \psi_{\tau + \varepsilon}^* O_\tau \psi_{\tau + \varepsilon} - \int dq \, \psi_\tau^* O_\tau \psi_\tau \right) \tag{544}$$

The Hamiltonian operator is responsible for evolving the wave function in laboratory time:

$$\psi_{\tau + e} = \exp(-\imath \varepsilon H)\psi_\tau \approx (1 - \imath \varepsilon H)\psi_\tau \tag{545}$$

By taking advantage of the fact that the Hamiltonian is Hermitian:

$$\frac{d \langle O \rangle}{d\tau} = \frac{1}{\varepsilon} \left(\int dq \, \psi_\tau^* (1 + \imath \varepsilon H) \left(O + \varepsilon \frac{\partial O}{\partial \tau} \right)(1 - \imath \varepsilon H)\psi_\tau - \psi_\tau^* O \psi_\tau \right) \tag{546}$$

and taking the limit as $\epsilon \to 0$ we get:

$$\frac{d \langle O \rangle}{d\tau} = -\imath \langle [O, H] \rangle + \left\langle \frac{\partial O}{\partial \tau} \right\rangle \tag{547}$$

This simplifies when the operators are not dependent on laboratory time:

$$\frac{\partial O}{\partial \tau} = 0 \to \frac{d \langle O \rangle}{d\tau} = -\imath [O, H] \tag{548}$$

This always the case for the operators of interest here. We assume that the clock time relates only to the observer, not to the system observed!

With the operator assignments above we have commutation rules:

$$[x_\mu, p_\nu] = -\imath g_{\mu\nu} \qquad (549)$$

or breaking this out:

$$\begin{aligned}[t, E] &= -\imath \\ [x_j, p_k] &= \imath \delta_{jk}\end{aligned} \qquad (550)$$

If we apply this rule to the coordinate and momentum operators we get for the coordinates:

$$\dot{t} = -\imath \left[t, -\frac{(p - eA)^2 - m^2}{2m} \right] = \frac{E - e\Phi}{m} \qquad (551)$$

$$\dot{\vec{x}} = -\imath \left[\vec{x}, -\frac{(p - eA)^2 - m^2}{2m} \right] = \frac{\vec{p} - e\vec{A}}{m} \qquad (552)$$

which are the classical equations of motion (B), as required.

D   Unitarity

Since there is a chance of the wave function "sneaking past" the plane of the present, we have to be particularly careful to confirm unitarity.

To establish that the path integral kernel is unitary we need to establish that it preserves the normalization of the wave function. The analysis in the text only established this for the free case. We therefore need to confirm that the normalization of the wave function is preserved in the general case. We use a proof from Merzbacher Merzbacher [1998] but in four rather than three dimensions.

We form the probability:

$$P \equiv \int d^4 x \psi^* (x) \psi (x) \qquad (553)$$

We therefore have for the rate of change of probability in time:

$$\frac{dP}{d\tau} = \int d^4 x \left(\psi^* (x) \frac{\partial \psi (x)}{\partial \tau} + \frac{\partial \psi^* (x)}{\partial \tau} \psi (x) \right) \qquad (554)$$

The Schrödinger equations for the wave function and its complex conjugate are:

$$\frac{\partial \psi}{\partial \tau} = -\frac{\imath}{2m} \partial^\mu \partial_\mu \psi + \frac{q}{m} (A^\mu \partial_\mu) \psi + \frac{q}{2m} (\partial^\mu A\mu) \psi + \imath \frac{q^2}{2m} A^\mu A_\mu \psi - \imath \frac{m}{2} \psi \quad (555)$$

$$\frac{\partial \psi^*}{\partial \tau} = \frac{\imath}{2m} \partial^\mu \partial_\mu \psi^* + \frac{q}{m} (A^\mu \partial_\mu) \psi^* + \frac{q}{2m} (\partial^\mu A\mu) \psi^* - \imath \frac{q^2}{2m} A^\mu A_\mu \psi^* + \imath \frac{m}{2} \psi^*$$
$$(556)$$

We rewrite $\frac{\partial\psi}{\partial\tau}$ and $\frac{\partial\psi^*}{\partial\tau}$ using these and throw out canceling terms. Since the probability density is gauge independent, we choose the Lorentz gauge $\partial A = 0$ to get:

$$\frac{dP}{d\tau} = \int d^4x \left(\psi^* \left(-\frac{\imath}{2m}\partial^\mu\partial_\mu\psi + \frac{q}{m}\left(A\partial\right)\psi \right) + \left(\frac{\imath}{2m}\partial^\mu\partial_\mu\psi^* + \frac{q}{m}\left(A\partial\right)\psi^* \right)\psi \right)$$

(557)

We integrate by parts; we are left with zero on the right:

$$\frac{dP}{d\tau} = 0 \qquad (558)$$

Therefore the rate of change of probability is zero, as was to be shown. And therefore the normalization is correct in the general case.

Note this is a special case of Ehrenfest's theorem in TQM with the probability P represented by the operator the number 1 (possibly the simplest operator in quantum mechanics):

$$\frac{d\langle 1\rangle}{d\tau} = -\imath\langle[1,H]\rangle + \left\langle \frac{\partial 1}{\partial\tau} \right\rangle = 0 \qquad (559)$$

where $-\imath H = \frac{\partial}{\partial\tau}$ and the rest as above.

E Gauge transformations

As noted in the text, the ambiguities in the normalization of the wave function may be seen as representing a kind of gauge transformation. We have all the usual possibilities for gauge transformations. And we have in addition the possibility of gauge transformations which are a function of the laboratory time.

To explore this, we write the wave function as a product of a gauge function in coordinate time, space, and laboratory time and a gauged wave function:

$$\psi'_\tau\left(t,\vec{x}\right) = e^{\imath q\Lambda_\tau\left(t,\vec{x}\right)}\psi_\tau\left(t,\vec{x}\right) \qquad (560)$$

If the original wave function satisfies a gauged Schrödinger equation:

$$\left(\imath\frac{\partial}{\partial\tau} - q\mathcal{A}_\tau\left(x\right) \right)\psi_\tau\left(x\right) = -\frac{1}{2m}\left(\left(p - qA\right)^2 - m^2 \right)\psi_\tau\left(x\right) \qquad (561)$$

the gauged wave function also satisfies a gauged Schrödinger equation:

$$\left(\imath\frac{\partial}{\partial\tau} - q\mathcal{A}'_\tau\left(x\right) \right)\psi'_\tau\left(x\right) = -\frac{1}{2m}\left(\left(p - qA'\right)^2 - m^2 \right)\psi'_\tau\left(x\right) \qquad (562)$$

provided we have:

$$\mathcal{A}'_\tau\left(x\right) = \mathcal{A}_\tau\left(x\right) - \frac{\partial\Lambda_\tau\left(x\right)}{\partial\tau} \qquad (563)$$

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and the usual gauge transformations:

$$A'^{\mu} = A^{\mu} - \partial^{\mu} \Lambda_{\tau}(x) \tag{564}$$

or:

$$\begin{aligned} \Phi' &= \Phi - \frac{\partial \Lambda}{\partial t} \\ \vec{A}' &= \vec{A} + \nabla \Lambda \end{aligned} \tag{565}$$

If the gauge function Λ is not a function of the laboratory time $(\Lambda = \Lambda(t, \vec{x}))$ then we recover the usual gauge transformations for Φ and $\vec{A}$. On the other hand, we could let the gauge depend on the laboratory time, perhaps using different gauges for different parts of the problem in hand.

F   Free wave functions and kernels

We here pull together many of the formulas for the free case for reference.

In general the free wave functions and kernels can be written as a coordinate time part times a familiar non-relativistic part. The division into coordinate time, space, and – occasionally – clock time parts is to some extent arbitrary.

F.1   Plane waves

Plane wave in coordinate time:

$$\tilde{\phi}_{\tau}(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\imath E_0 t + \imath \frac{E_0^2}{2m}\tau\right) \tag{566}$$

Plane wave in space:

$$\bar{\phi}_{\tau}(\vec{x}) = \frac{1}{\sqrt{2\pi}^3} \exp\left(\imath \vec{p_0} \cdot \vec{x} - \imath \frac{\vec{p_0}^2}{2m}\tau\right) \tag{567}$$

The full plane wave is the product of coordinate time and space plane waves:

$$\phi_{\tau}(x) = \tilde{\phi}_{\tau}(t)\,\bar{\phi}_{\tau}(\vec{x}) \exp\left(\imath \frac{m^2}{2m}\tau\right) = \frac{1}{4\pi^2} \exp\left(-\imath p_0 x - \imath f_0 \tau\right) \tag{568}$$

with definition of clock frequency:

$$f_0 \equiv -\frac{E_0^2 - \vec{p_0}^2 - m^2}{2m} \tag{569}$$

The equivalents in momentum space are δ functions with a clock time dependent phase:

$$\hat{\tilde{\phi}}_\tau (E) = \delta (E - E_0) \exp \left(\imath \tfrac{E_0^2}{2m} \tau \right)$$
$$\hat{\tilde{\phi}}_\tau (\vec{p}) = \delta^{(3)} (\vec{p} - \vec{p}_0) \exp \left(-\imath \tfrac{p_0^2}{2m} \tau \right) \qquad (570)$$
$$\hat{\tilde{\phi}}_\tau (p) = \hat{\tilde{\phi}}_0 (E) \hat{\tilde{\phi}}_0 (\vec{p}) \exp \left(-\imath \tfrac{m^2}{2m} \tau \right) = \delta^{(4)} (p - p_0) \exp \left(-\imath f_0 \tau \right)$$

F.2 Gaussian test functions

By Morlet wavelet decomposition any normalizable wave function may be written as a sum over Gaussian test functions. We have specified the conventions we are using for Gaussian test functions above; here we look specifically at Gaussian test functions as solutions of the free Schrödinger equation. There will be some overlap with the previous.

F.2.1 Time and energy

Gaussian test function in coordinate time at clock time zero:

$$\tilde{\varphi}_0 (t) \equiv \sqrt[4]{\frac{1}{\pi \sigma_t^2}} e^{-\imath E_0 (t - t_0) - \frac{(t - t_0)^2}{2\sigma_t^2}} \qquad (571)$$

$$\hat{\tilde{\varphi}}_0 (E) \equiv \sqrt[4]{\frac{1}{\pi \sigma_E^2}} e^{\imath E t_0 - \frac{(E - E_0)^2}{2\sigma_E^2}} \qquad (572)$$

With these conventions, the energy and coordinate time dispersions are reciprocals:

$$\sigma_E = \frac{1}{\sigma_t} \qquad (573)$$

Here we are using the "0" to label the constants in the wave function, to indicate they are the ones at starting time $\tau_0 = 0$. It is often convenient to thread a letter through the wave function to label the constants, e.g.:

$$\tilde{\varphi}_a (t) \equiv \sqrt[4]{\frac{1}{\pi \sigma_t^{(a)2}}} e^{-\imath E_a (t - t_a) - \frac{(t - t_a)^2}{2\sigma_t^{(a)2}}} \qquad (574)$$

$$\hat{\tilde{\varphi}}_a (E) \equiv \sqrt[4]{\frac{1}{\pi \sigma_E^{(a)2}}} e^{\imath E t_a - \frac{(E - E_a)^2}{2\sigma_E^{(a)2}}} \qquad (575)$$

Gaussian test function for coordinate time as a function of clock time:

$$\tilde{\varphi}_\tau (t) = \sqrt[4]{\frac{1}{\pi \sigma_t^2}} \sqrt{\frac{1}{f_\tau^{(t)}}} e^{-\imath E_0 t - \imath \frac{E_0^2}{2m} \tau - \frac{1}{2\sigma_t^2 f_\tau^{(t)}} \left(t - t_0 - \frac{E_0}{m} \tau \right)^2} \qquad (576)$$

with dispersion factor $f_\tau^{(t)} \equiv 1 - \imath \frac{\tau}{m \sigma_t^2}$.

and with expectation, probability density, and uncertainty ($\gamma \equiv \frac{E}{m}$):

$$\langle t \rangle = t_0 + \frac{E}{m}\tau = t_0 + v_0\tau = t_0 + \gamma\tau$$

$$\tilde{\rho}_\tau(t) = \sqrt{\frac{1}{\pi\sigma_t^2}}\exp\left(-\frac{(t-\gamma\tau)^2}{\sigma_t^2\left(1+\frac{\tau^2}{m^2\sigma_t^2}\right)}\right) \tag{577}$$

$$(\Delta t)^2 \equiv \langle t^2 \rangle - \langle t \rangle^2 = \frac{\sigma_t^2}{2}\left|1 + \frac{\tau^2}{m^2\sigma_t^4}\right|$$

In the non-relativistic case, if we start with $t_0 = \tau$, then we have $\langle t \rangle \approx \tau$ throughout.

For longer clock times the uncertainty in coordinate time is proportional to the uncertainty in the energy:

$$\Delta t \sim \frac{\tau}{m\sigma_t} = \frac{\tau}{m}\sigma_E \tag{578}$$

Gaussian test function in energy:

$$\hat{\tilde{\varphi}}_\tau(E) \equiv \sqrt[4]{\frac{1}{\pi\sigma_E^2}}e^{\imath E t_0 - \frac{(E-E_0)^2}{2\sigma_E^2} + \imath\frac{E^2}{2m}\tau} \tag{579}$$

with expectation, probability density, and uncertainty:

$$\langle E \rangle = E_0$$

$$\hat{\tilde{\rho}}_\tau(E) = \hat{\tilde{\rho}}_0(E) = \sqrt{\frac{1}{\pi\sigma_E^2}}\exp\left(-\frac{(E-E_0)^2}{\sigma_E^2}\right) \tag{580}$$

$$(\Delta E)^2 = \frac{\sigma_E^2}{2}$$

F.2.2 Single space/momentum dimension

Gaussian test function in one space dimension at clock time zero:

$$\bar{\varphi}_0(x) = \sqrt[4]{\frac{1}{\pi\sigma_x^2}}e^{\imath p_0(x-x_0) - \frac{(x-x_0)^2}{2\sigma_x^2}} \tag{581}$$

and in momentum:

$$\hat{\bar{\varphi}}_0(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}}e^{-\imath p x_0 - \frac{(p-p_0)^2}{2\sigma_p^2}} \tag{582}$$

The space and momentum dispersions are reciprocal:

$$\sigma_p \equiv \frac{1}{\sigma_x} \tag{583}$$

When we have to consider the dispersion for all three space momentum we reduce the level of nesting by writing:

$$\hat{\sigma}_x \equiv \sigma_{p_x}, \hat{\sigma}_y \equiv \sigma_{p_y}, \hat{\sigma}_z \equiv \sigma_{p_z} \tag{584}$$

Gaussian test function in one space dimension as a function of clock time:

$$\bar{\varphi}_\tau(x) = \sqrt[4]{\frac{1}{\pi\sigma_x^2}} \sqrt{\frac{1}{f_\tau^{(x)}}} e^{\imath p_0 x - \frac{1}{2\sigma_x^2 f_\tau^{(x)}} \left(x - x_0 - \frac{p_0}{m}\tau\right)^2 - \imath\frac{p_0^2}{2m}\tau} \tag{585}$$

The definition of the dispersion factor $f_\tau^{(x)} = 1 + \imath\frac{\tau}{m\sigma_x^2}\tau$ is parallel to that for coordinate time (but with the opposite sign for the imaginary part).

Expectation, probability density, and uncertainty for x:

$$\begin{aligned}
\langle x \rangle &= x_0 + \frac{p_x}{m}\tau = x_0 + v_x\tau \\
\bar{\rho}_\tau(x) &= \sqrt{\frac{1}{\pi\sigma_1^2}} \exp\left(-\frac{(x - \langle x_\tau \rangle)^2}{\sigma_x^2}\right) \\
(\Delta x)^2 &\equiv \langle x_\tau^2 \rangle - \langle x_\tau \rangle^2 = \frac{\sigma_x^2}{2}\left|1 + \frac{\tau^2}{m^2\sigma_x^4}\right|
\end{aligned} \tag{586}$$

and similarly for y and z.

As clock time goes to infinity, the dispersion in space scales as:

$$(\Delta x)^2 \approx \frac{\bar{\tau}^2}{2m^2\sigma_x^2} \tag{587}$$

Negative x-momentum is movement to the left, positive to the right. As we require the most complete parallelism between time and space, we therefore have that positive energy corresponds to movement into the future, negative into the past. As most of our wave functions have an energy of order:

$$E \sim m + \frac{\bar{p}^2}{2m} \gg 0 \tag{588}$$

they are usually going into the future. As expected.

Gaussian test function for momentum in one dimension as a function of clock time:

$$\hat{\bar{\varphi}}_\tau(p) = \sqrt[4]{\frac{1}{\pi\sigma_p^2}} e^{-\imath p x_0 - \frac{(p - p_0)^2}{2\sigma_p^2} - \imath\frac{p^2}{2m}\tau} \tag{589}$$

and expectation, probability density, and uncertainty for p:

$$\begin{aligned}
\langle p \rangle_\tau &= \langle p \rangle_0 = p_0 \\
\hat{\bar{\rho}}_\tau(p) &= \hat{\bar{\rho}}_0(p) = \sqrt{\frac{1}{\pi\sigma_p^2}} \exp\left(-\frac{(p - p_0)^2}{\sigma_p^2}\right) \\
(\Delta p)^2 &= \frac{\sigma_p^2}{2}
\end{aligned} \tag{590}$$

F.2.3   Covariant forms

Usually we treat the case of four dimensions as a simple product of the cases of the individual dimensions. But it is more appropriate in general to treat them as a single covariant object.

We define the four dimensional dispersion Σ in position space at clock time zero:

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$$\Sigma_0^{\mu\nu} \equiv \begin{pmatrix} \sigma_t^2 & 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_z^2 \end{pmatrix} \tag{591}$$

The determinant takes a simple form:

$$\det \Sigma = \sigma_t^2 \sigma_x^2 \sigma_y^2 \sigma_z^2 \tag{592}$$

With this we have the wave function at clock time zero:

$$\varphi_0 (x) = \sqrt[4]{\frac{1}{\pi^4 \det (\Sigma_0)}} e^{-\imath p_0^\mu (x-x_0)_\mu - \frac{1}{2}(x-x_0)_\mu \Sigma_\tau^{\mu\nu} (x-x_0)_\nu} \tag{593}$$

Four dimensional dispersion as a function of clock time τ:

$$\Sigma_\tau^{\mu\nu} \equiv \begin{pmatrix} \sigma_t^2 f_\tau^{(t)} = \sigma_t^2 - \imath \frac{\tau}{m} & 0 & 0 & 0 \\ 0 & \sigma_x^2 f_\tau^{(x)} = \sigma_x^2 + \imath \frac{\tau}{m} & 0 & 0 \\ 0 & 0 & \sigma_y^2 f_\tau^{(y)} = \sigma_y^2 + \imath \frac{\tau}{m} & 0 \\ 0 & 0 & 0 & \sigma_z^2 f_\tau^{(z)} = \sigma_z^2 + \imath \frac{\tau}{m} \end{pmatrix} \tag{594}$$

Four dimensional wave function as a function of clock time:

$$\varphi_\tau (x) = \sqrt[4]{\frac{\det (\Sigma_\tau)}{\pi^4 \det^2 (\Sigma_\tau)}} e^{-\imath p_0^\mu (x-x_0-v\tau)_\mu - \frac{1}{2}(x-x_0-v\tau)_\mu \Sigma_\tau^{\mu\nu} (x-x_0-v\tau)_\nu - \imath f_0 \tau} \tag{595}$$

with the obvious definition of the four velocity:

$$v_\mu \equiv \frac{p_\mu}{m} \tag{596}$$

In the general case, the wave function does not, of course, split into coordinate time and space factors. But it may still be represented as a sum over basis wave functions that do, via Morlet wavelet decomposition.

We have similar but simpler formulas in momentum space. In momentum space at clock time zero:

$$\hat{\varphi}_0 (p) = \sqrt[4]{\frac{1}{\pi^4 \det \left(\hat{\Sigma}_0\right)}} e^{-\imath p x_0 - \frac{1}{2}(p-\hat{p_0})_\mu \Sigma^{\mu\nu} (p-p_0)_\nu} \tag{597}$$

and as a function of clock time:

$$\hat{\varphi}_\tau (p) = \sqrt[4]{\frac{1}{\pi \det^2 \left(\hat{\Sigma}_0\right)}} e^{-\imath p x_0 - \frac{1}{2}(p-\hat{p_0})_\mu \Sigma^{\mu\nu} (p-p_0)_\nu - \imath \omega_0 \tau} = \hat{\varphi}_0 (p) \exp (-\imath f_0 \tau) \tag{598}$$

We define the momentum dispersion matrix $\hat{\Sigma}$ as the reciprocal of the coordinate space dispersion matrix Σ_0:

$$\hat{\Sigma} \equiv \begin{pmatrix} \hat{\sigma}_t^2 & 0 & 0 & 0 \\ 0 & \hat{\sigma}_x^2 & 0 & 0 \\ 0 & 0 & \hat{\sigma}_y^2 & 0 \\ 0 & 0 & 0 & \hat{\sigma}_z^2 \end{pmatrix} \tag{599}$$

F.3 Free kernels

We list the kernels corresponding to the free Schrödinger equation in time. These are retarded kernels going from clock time zero to clock time τ, so include an implicit $\theta(\tau)$.

F.3.1 Coordinate time and space

Kernel in coordinate time:

$$\tilde{K}_\tau(t''; t') = \sqrt{\frac{\imath m}{2\pi\tau}} \exp\left(-\imath m \frac{(t'' - t')^2}{2\tau}\right) \tag{600}$$

In three space we have the familiar non-relativistic kernel (e.g. Merzbacher [1998]):

$$\bar{K}_\tau(\vec{x}''; \vec{x}') = \sqrt{-\frac{\imath}{2\pi\tau}}^{\,3} \exp\left(\imath m \frac{(\vec{x}'' - \vec{x}')^2}{2\tau}\right) \tag{601}$$

The full kernel is:

$$K_\tau(x''; x') = \tilde{K}_\tau(t''; t') \bar{K}_\tau(\vec{x}''; \vec{x}') \exp\left(-\imath \frac{m}{2}\tau\right) \tag{602}$$

Explicitly:

$$K_\tau(x''; x') = -\imath \frac{m^2}{4\pi^2\tau^2} e^{-\imath m \frac{(t'' - t')^2}{2\tau} + \imath m \frac{(\vec{x}'' - \vec{x}')^2}{2\tau} - \imath \frac{m}{2}\tau} \tag{603}$$

F.3.2 Momentum space

Energy part:

$$\hat{\tilde{K}}_\tau(E''; E') = \delta(E'' - E') \exp\left(\imath \frac{E'^2 - m^2}{2m}\tau\right) \tag{604}$$

Three momentum part:

$$\hat{\vec{K}}_\tau(\vec{p}''; \vec{p}') = \delta^{(3)}(\vec{p}'' - \vec{p}') \exp\left(-\imath \frac{(\vec{p}')^2}{2m}\tau\right) \tag{605}$$

Again, the same as the usual non-relativistic kernel.

The full kernel is:

$$\hat{K}_\tau \left(p'' - p'\right) = \hat{\tilde{K}}_\tau \left(E''; E'\right) \hat{\vec{K}}_\tau \left(\vec{p}''; \vec{p}'\right) \tag{606}$$

Spelled out:

$$\hat{K}_\tau \left(p''; p'\right) = \delta^{(4)} \left(p'' - p'\right) \exp\left(\imath \frac{E'^2 - (\vec{p}')^2 - m^2}{2m}\tau\right) \tag{607}$$

or:

$$\hat{K}_\tau \left(p''; p'\right) = \delta^{(4)} \left(p'' - p'\right) \exp\left(-\imath f_{p'}\tau\right) \tag{608}$$

With definition of clock frequency as above:

$$f_p \equiv -\frac{E^2 - \vec{p}^2 - m^2}{2m} \tag{609}$$
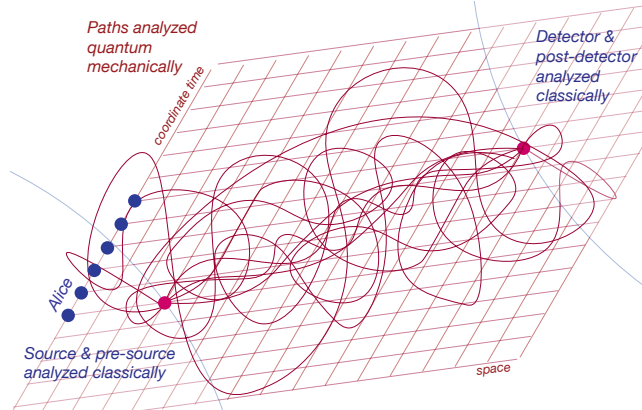
G Checkpoint Copenhagen

Figure 19: Clock time and coordinate time

"Atoms are completely impossible from the classical point of view, since the electrons would spiral into the nucleus." – Richard P. Feynman Feynman et al. [1965]

G.1 Four dimensions and an approximation scheme

Naively we might appear to have a five dimensional coordinate system here: clock time, coordinate time, and the three space dimensions. When this work was presented at the 2018 International Assocation for Relativistic Dynamics

(IARD) conference Dr. Asher Yahalom observed that this is potentially a bit cumbersome.

Per subsequent discussion with Dr. Yahalom, it is more accurate to describe it as four dimensions plus an approximation scheme.

Let's start with a sheet of four dimensional graph paper representing discretized space time. Alice is off to one side, drawing paths on it. She is interested in the amplitude for a particle to get from A to B. And to compute this she draws all possible paths from A to B, planning to sum them using the rules in the text.

But Alice herself is a part of the universe she is observing.

Therefore we draw Alice on the left as a series of blue dots representing her at successive clock ticks. At each clock tick she has a corresponding three dimensional hyper-surface representing her rest frame. These are the horizontal lines in the diagram. She is off to one side, because she is after all observing the particle going from A to B, but she is on the same piece of four dimensional graph paper because she is part of the same universe.

If she is using SQM to do the calculation then at each clock tick each of her paths will slide side to side on the corresponding hyper-surface. But if she is using TQM then the paths will also go forwards and backwards in time, going off the current hyper-surface, often in quite elaborate ways.

If Bob is also present, we can represent him by a series of green dots on the right with his own coordinate system. (We have left this off the illustration because it would make it too busy. Please imagine Bob's green dots and hyper-surfaces are present.) He will be looking at the same set of paths, but slicing them up differently because he has in general a different set of three dimensional hyper-surfaces.

And if we need to resolve the slight differences between Alice and Bob's descriptions – per the discussion in the text (subsection 3.5) – we can call in Vera in the V frame (violet dots of course) to provide the definitive story.

So far so good, we have only one four dimensional coordinate system with different observers. Each observer has his or her clock time, but these clock times are present, tick by tick, on the same piece of graph paper. And we have well-defined rules for going from one observer's frame to another's.

The problem comes in when we try to reconcile the quantum descriptions Alice uses for particles with the classical descriptions she uses for detectors, emitters, or herself. We have an impedance mismatch between quantum and classical descriptions. This is of course the problem of measurement.

G.2   Quantum descriptions and classical approximations

The critical observation here is Feynman's: there are no classical atoms. Since the emitters, detectors, and observers are all made of atoms all are quantum objects.

This means that there is no transition from a quantum to a classical realm. Everything on both sides of the act of detection is a quantum system.

So what is going on here?

Most of the systems we deal with can treated for all practical purposes (FAPP) as if they were classical systems. It is only when we look at certain parts of the system that we need to get down to the quantum level. When we describe a particle being ejected from an atom we need to use a quantum description. When we describe a particle in flight we need to use a quantum description. When we describe its encounter with a detector we need to use a quantum description. But once the detector has registered a click, we can use classical approaches to describe the counting and processing of those clicks.

The rules for detection and emission are ways to navigate the associated approximations. In a detector, the wave function is not collapsing, instead we are passing from a quantum to a classical description. And in emission, the particle is not originally classical, it is just that up to the starting gun, it can be treated as if it were.

In human terms, picture Alice going thru passport control. Bob, now a customs official (he gets around), stamps her passport with a visa stamp. She then heads to her ultimate destination. Both Alice and Bob are – per Feynman – quantum systems. Their previous and subsequent paths are unknown to official customs. But we have that visa stamp and associated computer records. They are the measurement. They certify that at time T position X with uncertainties ΔT and ΔX Alice and Bob encountered each other.

Since Bob's location is highly localized – he is in a customs booth, they are not big – we treat this as a measurement of Alice's position. But the situation is in reality completely symmetrical. We can think of it as Alice measuring Bob or Bob measuring Alice. But for customs purposes only the visa stamp matters. The visa stamp is the measurement.

Ultimately we must always be prepared to go down to the quantum level. The quantum rules are decisive; the classical rules a mere useful approximation. Even the visa stamp itself is made of atoms, of quantum mechanical objects. But in some cases we can get away with a classical analysis.

The division between quantum and classical is a division of analysis. It is not part of the physical universe, it is part of how we describe that universe – allowing for the fact that we are a part of what we are describing.

G.3 Where, when, and to what extent?

But to take this point from the realm of philosophy to the realm of science we need to throw some numbers into the mix, we need to answer the questions: "when, where, and to what extent does the classical approximation break down?"

The rules in the Copenhagen interpretation are not specific. "Somewhere" between where the particle is being described by quantum mechanics and where it is detected, the particle goes through a "checkpoint Copenhagen", where its description collapses from the fuzziness of quantum mechanics to the determinism and specificity of classical mechanics.

This is both a strength and a weakness of the Copenhagen interpretation.

It is a strength because it let physicists get on with physics. And because it does not try to "explain" what is going on.

But it is a weakness because the terms of the transition are not specified. Schrödinger's cat experiment is the most striking illustration of this point Schrödinger [1980].

Decoherence provides part of the answer Omnès [1994], Giulini et al. [1996], Heiss [2002], Joos [2003], Schlosshauer [2007]. But decoherence is not yet as quantitative as one might like (although see Venugopalan, Qureshi, and Mishra Venugopalan et al. [2019]).

It is possible that new physics plays a role in the transition. The Ghirardi, Rimini, and Weber (GRW) Ghirardi et al. [1986]approach and more generally the continuous spontaneous localization (CSL) approaches Dickson [1998] hypothesize additional physics. These alternatives are helpful in parameterizing the transition, but have not had any experimental confirmation.

Meanwhile, the quantum cats continue to get bigger and bigger, less and less microscopic Norte et al. [2016], Vinante et al. [2017]. At some point we will actually be able to see the transition itself, or show that there is none.

TQM does *not* specifically address this question. We have taken the existing rules as given. However the temporal fluctuations in TQM (see especially the entropic estimate of the initial wave function 4.1.2) provide a source of "internal decoherence" so would affect estimates of the size and rate of decoherence. (We owe the phrase "internal decoherence" to Dr. Daniel Braun at the 2007 Feynman Festival.)

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